89 Loop Quantum Cosmology: Physics of Singularity Resolution and its Implications

89 圈量子宇宙学: 奇点消解的物理及其启示

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Abstract

摘要

The occurrences of singularities where spacetime curvature becomes infinite and geodesic evolution breaks down are inevitable events in classical general relativity (GR) unless one chooses an exotic matter violating weak energy condition. These singularities show up in various physical processes, such as the gravitational collapse, the birth of the universe in the standard cosmology as well as the classical solutions of the black hole spacetimes. In the last two decades, a rigorous understanding of the dynamics of quantum spacetime and the way it resolves singularities has been achieved in loop quantum cosmology (LQC) which applies the concepts and techniques of loop quantum gravity to the symmetry reduced cosmological spacetimes. Due

to the fundamental discreteness of quantum geometry derived from the quantum theory, the big-bang singularity has been robustly shown to be replaced by a big bounce. Strong curvature singularities intrinsic in the classical cosmology are generically resolved for a variety of cosmological spacetimes including anisotropic models and polarized Gowdy models. Using effective spacetime description, the LQC universe also provides an ultra-violet complete description of the classical inflationary scenario as well as its alternatives such as the ekpyrotic and matter-bounce scenarios. In this chapter, we provide a summary of singularity resolution and its physical implications for various isotropic and anisotropic cosmological spacetimes in LQC and analyze robustness of results through variant models originating from different quantization prescriptions.

在经典广义相对论 (GR) 中，时空曲率趋于无穷、测地线演化终止的奇点必然会出现，除非选择违背弱能量条件的奇异物质。这类奇点出现在多种物理过程中，例如引力坍缩、标准宇宙学中的宇宙创生，以及黑洞时空的经典解。近二十年来，将圈量子引力的概念和技术应用于对称性约化的宇宙学时空中得到的圈量子宇宙学 (LQC)， 已经实现了对量子时空动力学及其解决奇点方式的严格认知。得益于该量子理论导出的量子几何的基本离散性，已有充分证据表明大爆炸奇点被大反弹取代。对于包括各向异性模型和极化 Gowdy 模型在内的多种宇宙学时空中，经典宇宙固有的强曲率奇点一般都能得到解决。借助有效时空描述，圈量子宇宙学中的宇宙也能对经典暴胀场景，以及火劫场景、物质反弹场景等替代方案给出紫外完备描述。本章我们总结了圈量子宇宙学中各类各向同性、各向异性宇宙学时空中的奇点 resolution 及其物理意义，并通过不同量子化方案衍生的各类模型分析了结果的稳健性。

Keywords

关键词

Loop quantum cosmology - Singularity resolution - Effective dynamics . Anisotropic models - Inflationary paradigm - Ekpyrotic scenario .

圈量子宇宙学——奇点消除——有效动力学。各向异性模型——暴胀范式——火劫场景。

Matter-bounce scenario

物质反弹场景

Introduction

引言

Theorems of Penrose, Hawking and Geroch show that singularities, such as the big bang, are the generic features of spacetimes in general relativity (GR) [1-3]. These are the boundaries of spacetime at which the classical evolution stops and the known laws of physics reach the limit of their validity. A new theory is needed to go beyond these final boundaries of classical spacetime to make definitive predictions about the initial state of our universe, to understand the emergence of the classical space and time, and to explore the new physics of the very early universe, and deep inside the black hole interiors. It has been long believed that an understanding of quantum gravitational effects would provide important insights on the resolution of singularities. One such avenue, where quantum gravity effects have been used to understand the problem of singularities, is loop quantum cosmology (LQC) [4]. It is a non-perturbative canonical quantization of homogeneous spacetimes using the techniques of loop quantum gravity (LQG) [5-7], a candidate theory of quantum gravity based on Ashtekar-Barbero variables. Though LQG is not yet a complete theory, in the last three decades sufficient mathematical control has been achieved to use it to quantize spacetimes with reduced degrees of freedom, such as cosmological and black hole spacetimes. A key prediction of LQG is that geometrical operators have discrete eigenvalues, at least at the kinematical level. However, due to its complexity, it becomes much easier and more straightforward to test the technical and conceptual issues of LQG in the context of symmetry reduced spacetimes where great simplifications resulting from symmetry reduction lead to a manageable quantum theory. Taking into account the influx of the observational data of the early universe with increasing precision in the recent years, cosmology provides an ideal platform where predictions from the quantum theory are expected to be tested by direct observational signals. In this setting, LQC provides a novel for quantizing cosmological spacetimes using the infrastructure of LQG.

彭罗斯、霍金与杰罗奇定理表明，在广义相对论 (GR) 中，大爆炸这类奇点是时空的普遍特征 [1-3]。这些奇点是时空的边界: 经典演化在此终止，已知物理定律也在此达到其适用极限。我们需要新理论突破经典时空的这些终极边界，对宇宙的初态给出确定预言，理解经典时空的涌现，探索极早期宇宙以及黑洞内部深处的新物理。长期以来，人们认为理解量子引力效应会为奇点消解问题提供重要洞见。利用量子引力效应研究奇点问题的一个方向是圈量子宇宙学 (LQC)[4]。它是借助圈量子引力 (LQG) 技术对均匀时空完成的非微扰正则量子化 [5-7]；而 LQG 是基于阿西特卡-巴贝罗变量的量子引力候选理论。尽管 LQG 尚未成为完备理论，过去三十年间我们已经获得了足够的数学控制度，可利用它对自由度约化后的时空 (例如宇宙学时空与黑洞时空) 进行量子化。LQG 的一个关键预言是: 几何算符至少在运动学层面具有离散本征值。然而受其复杂性影响，在对称性约化时空的框架下检验 LQG 的技术与概念问题会更简单直接，因为对称性约化带来的大幅简化使量子理论易于处理。近年来精度不断提升的早期宇宙观测数据大量涌现，宇宙学成了理想平台——人们有望通过直接观测信号检验量子理论的预言。在此背景下，LQC 为借助 LQG 的基础框架量子化宇宙学时空提供了全新方法。

LQC inherits discrete quantum geometry from LQG. This results in significant qualitative differences between LQC and earlier attempts to quantum cosmology, such as the Wheeler-DeWitt theory. These qualitative differences are most prominent only when spacetime curvature is large and comparable to the Planck scale. At small spacetime curvatures, classical differential geometry turns out to be an excellent approximation of the discrete quantum geometry of LQC, and GR is recovered. LQC thus passes an important test required for any theory of quantum cosmology/gravity-it has the correct low energy limit. Since LQC is a canonical quantization, dynamics is governed by a Hamiltonian constraint, which due to the underlying quantum geometry turns out to be quantum difference equation. The structure of the Hamiltonian constraint in LQC is in a striking contrast to the one in the Wheeler-DeWitt theory where dynamics is governed by a differential Wheeler-DeWitt equation. Unlike the Wheeler-DeWitt theory where the problem of singularities persists even after quantization, in LQC discrete quantum geometry results in a quantum bounce when spacetime curvature reaches Planck scale [8-10].

LQC 从 LQG 继承了离散量子几何。这让 LQC 与惠勒-德维特理论这类早期量子宇宙学尝试存在显著质的差异，这些质的差异只有在时空曲率大到与普朗克尺度相当的时候才会显现。在低时空曲率区域，经典微分几何是 LQC 离散量子几何的极佳近似，广义相对论也能得以恢复。因此 LQC 通过了任何量子宇宙学/量子引力理论都需要满足的一项重要检验——它拥有正确的低能极限。由于 LQC 是正则量子化，动力学由哈密顿约束支配；受底层量子几何的影响，该约束是一个量子差分方程。LQC 中哈密顿约束的结构与惠勒-德维特理论截然不同，后者的动力学由微分形式的惠勒-德维特方程支配。惠勒-德维特理论即使在量子化后奇点问题依然存在，而 LQC 中，当时空曲率达到普朗克尺度时，离散量子几何会引发量子反弹 [8-10]。

The first rigorous model of LQC with a physical Hilbert space was constructed for a spatially flat homogeneous and isotropic FLRW universe sourced with a massless scalar field [8-10]. Its quantum theory was formulated on the lines of Dirac quantization approach for the constrained systems. In comparison to the older quantum cosmology proposed initially by DeWitt [11], LQC has the following two notable properties. First, it utilizes a quantum representation based on the holonomies of the Ashtekar-Barbero connection and gauge-fixed triads which is unitarily inequivalent to the Schrödinger representation used in the Wheeler-DeWitt quantum cosmology. The second main difference arises from the underlying quantum geometry. In the geometric representation, the differential Wheeler-DeWitt equation is replaced by a quantum difference equation where the difference in geometric sector is determined by the quantum geometry. The Hamiltonian constraint yields a Schrödinger-like equation, providing a dynamical evolution of physical states, with temporal role played by the massless scalar field. While the latter is not discretized, the evolution generator captures the underlying quantum geometry and is a discrete operator. In particular, the minimal nonzero eigenvalue of the area operator in LQG is incorporated in defining a nonlocal curvature operator [12], leading to the well-known $\overline{\mu}$ scheme in LQC [10]. Using sophisticated numerical simulations of the quantum difference equation [10,13,14], it has been rigorously shown that irrespective of the initial conditions, the big-bang singularity is resolved and replaced by a quantum bounce. The bounce takes place at a fixed maximum energy density in the Planck regime as is predicted by an exactly solvable model in LQC [15]. If one starts evolution of the universe when it is macroscopic and peaked on a classical GR trajectory at late times, it turns out that states remain sharply peaked across the bounce [16, 17] . On the other side of the bounce, there exists a contracting branch asymptoting to the same classical universe as in the expanding branch. The occurrence of the quantum bounce is a direct result of the minimal area gap of the loop over which holonomies are constructed. Using consistent histories formalism, one can show that the probability of bounce turns out to be unity [18]. In contrast, in the Wheeler-DeWitt theory, the expectation value of the volume operator follows the classical trajectories which inevitably end up with a vanishing volume signifying the encounter with the big-bang singularity [10]. Even if one considers an arbitrary superposition of expanding and contracting branches in the Wheeler-DeWitt theory, the consistent quantum probability for bounce is zero and for big bang is unity [19,20]. Therefore, two quantum theories, namely LQC and Wheeler-DeWitt theory, only converge in the classical limit. In the Planck regime, they lead to distinct physical predictions.

首个带有物理希尔伯特空间的严格 LQC 模型是针对无质量标量场源的空间平坦均匀各向同性 FLRW 宇宙构建的 [8-10]。其量子理论遵循狄拉克约束系统量子化方法构建。与 DeWitt 最初提出的旧量子宇宙学 [11] 相比，LQC 具有两个显著特性: 首先，它采用基于 Ashtekar-Barbero 联络全纯和规范固定三重态的量子表示，这与惠勒-德维特量子宇宙学使用的薛定谔表示是幺正不等价的。第二个核心差异源于基础量子几何: 在几何表示中，微分形式的惠勒-德维特方程被量子差分方程取代，几何部分的差分由量子几何决定。哈密顿约束给出类薛定谔方程，提供物理态的动力学演化，时间角色由无质量标量场承担。后者不离散化，但演化生成元承载基础量子几何，是一个离散算符。具体来说，LQG 面积算符的最小非零本征值被纳入非局域曲率算符的定义 [12]，由此得到 LQC 中著名的 $\bar{\mu}$ 方案 [10]。通过对量子差分方程的精细数值模拟 [10,13,14]，已严格证明: 无论初始条件如何，大爆炸奇点都会被消解，代之以量子反弹。正如 LQC 可精确求解模型预测的那样 [15]，反弹发生在普朗克区域的固定最大能量密度处。如果从宇宙宏观膨胀阶段、且物理态在经典广义相对论轨迹上呈峰值分布开始演化，结果会发现态在整个反弹过程始终保持尖锐峰值分布 [16, 17] 。在反弹的另一侧，存在一个收缩分支，它渐近于膨胀分支的同一个经典宇宙。量子反弹的发生是构造全纯的圈存在最小面积间隙的直接结果。利用一致性历史形式论可以证明，反弹发生的概率为 1[18]。与之相反，在惠勒-德维特理论中，体积算符的期望值遵循经典轨迹，不可避免地终止于零体积，对应大爆炸奇点的出现 [10]。即使在惠勒-德维特理论中考虑膨胀分支和收缩分支的任意叠加，反弹发生的一致量子概率为零，大爆炸发生的概率为 1[19,20]。因此，LQC 和惠勒-德维特这两种量子理论仅在经典极限下收敛，在普朗克区域它们给出截然不同的物理预言。

Interestingly, with the help of a complete set of the Dirac observables and the well-defined semiclassical physical states, the quantum evolution of the universe in LQC is shown to be endowed with a continuum spacetime description which is governed by an effective Hamiltonian constraint. This effective dynamics captures the leading-order corrections of the quantum theory and yields modified Friedmann and Raychaudhuri equations in LQC [21, 22]. The effective dynamics has been extensively used in the literature, and it plays an important role in demonstrating the robustness of the singularity resolution [23-28] and understanding the Planck-scale physics of the LQC universe extended from some well-known scenarios developed in the classical cosmology, such as the inflationary models [29-47], and ekpyrotic-and matter-bounce scenarios [48-53]. Extensive work has been performed in the last decade to understand quantum geometry effects on cosmological perturbations (see [54] for a review, and Chap. 90, "Loop Quantum Cosmology: Relation Between Theory and Observations").

有趣的是，借助完备的狄拉克可观测量集和定义良好的半经典物理态，已经证明 LQC 中宇宙的量子演化拥有由有效哈密顿约束支配的连续时空描述。该有效动力学 capture 了量子理论的领头阶修正，给出 LQC 中修正的弗里德曼方程和瑞查得符里方程 [21,22]。有效动力学已在文献中被广泛使用，它在证明奇点消解的鲁棒性 [23-28]、理解从经典宇宙学已有知名场景 (如暴涨模型 [29-47]、火劫模型和物质反弹场景 [48-53]) 拓展而来的 LQC 宇宙普朗克尺度物理中发挥着重要作用。过去十年，学界已开展大量工作研究量子几何效应对宇宙学扰动的影响 (综述见 [54]，另可参见第 90 章 "圈量子宇宙学: 理论与观测的联系")。

With the success achieved by LQC in yielding a physically viable theory of quantum cosmology in the simplest settings, the same techniques have been extended to loop quantize other cosmological spacetimes with more complicated structures, such as FLRW universe with spatial curvature [55-57], Bianchi universe with anisotropies and spatial curvature [58-61], and Gowdy models with continuous degrees of freedom [62-67]. The rigorous construction of the mathematical structure of the loop quantized cosmological spacetimes, including the kinematical Hilbert space, the Hamiltonian constraint operator, the Dirac observables as well as the physical inner product, leads to a consistent picture of singularity resolution and the Planck-scale physics in each model. Various unique properties of the particular quantum spacetimes have been studied in detail by using the effective dynamics, resulting in a deeper understanding of how the quantum geometry effects can change the classical description of the cosmological spacetimes. For example, from the modified Friedmann and Raychaudhuri equations, quantum geometry effects can be interpreted as making gravity repulsive in the Planck regime and resulting in a singularity resolution without any violation of any energy conditions.

随着 LQC 在最简框架下成功得到了物理上可行的量子宇宙学理论，相同的技术已被推广到对结构更复杂的其他宇宙学时空中进行圈量子化，例如具有空间曲率的 FLRW 宇宙 [55-57]、具有各向异性和空间曲率的 Bianchi 宇宙 [58-61]，以及具有连续自由度的 Gowdy 模型 [62-67]。对圈量子化宇宙学时空中数学结构的严格构造——包括运动学希尔伯特空间、哈密顿约束算符、狄拉克可观测量以及物理内积——在每个模型中都给出了奇点消解与普朗克尺度物理的自洽图像。人们已经利用有效动力学详细研究了特定量子时空的各类独特性质，从而更深入地理解量子几何效应如何改变宇宙学时空的经典描述。例如，从修正的弗里德曼方程和雷恰胡里方程可得，量子几何效应可被解释为: 引力在普朗克区域表现为排斥力，进而在不违反任何能量条件的前提下实现奇点消解。

Furthermore, attempts have been made in the direction of incorporating more features from LQG into the LQC universe. Since the loop quantization in LQC is implemented in the symmetry reduced mini-superspace, its relationship with the cosmological sector of the full theory is not yet established [68, 69], and ongoing research is being conducted in order to reach a better understanding of the essential features of the LQG cosmology. Recent progresses have been reported in the quantum-reduced loop quantum gravity [70], the group theory cosmology [71, 72], the path integral approach [73], and modified LQC models using the coherent states [74,75]. In particular, strong singularities are proved to be resolved and replaced by a quantum bounce as well in the modified LQC models [76], and moreover, richer structures beyond standard LQC, such as an asymmetric bounce and a pure quantum state in the contracting phase, etc. [77], have been observed, which imply that the cosmological sector from LQG can be much more complicated than expected. These new features of the modified LQC models have so far been studied only in the simplest spatially flat FLRW universe, and their impacts on other cosmological spacetimes with spatial curvature and anisotropies are still to be investigated.

此外，目前已有研究尝试将完整 LQG 的更多特征引入 LQC 宇宙中。由于 LQC 的圈量子化是在对称约化的微型超空间中实现的，它与完整理论宇宙学 sector 的关系尚未建立 [68, 69]，目前仍在开展研究以更好地理解 LQG 宇宙学的核心特征。已报道的最新进展包括量子约化圈量子引力 [70]、群论宇宙学 [71, 72]、路径积分方法 [73]，以及利用相干态构造的修正 LQC 模型 [74,75]。特别地，在修正 LQC 模型中也已证明强奇点会被消解并替换为量子反弹 [76]，此外还观测到了标准 LQC 之外更丰富的结构，例如非对称反弹、收缩相中的纯量子态等 [77]，这些结果表明 LQG 的宇宙学 sector 可能比预期复杂得多。到目前为止，修正 LQC 模型的这些新特征仅在最简单的空间平坦 FLRW 宇宙中得到研究，它们对具有空间曲率和各向异性的其他宇宙学时空中的影响仍有待探索。

In this chapter, we summarize some of the main results of LQC on the singularity resolution for various types of cosmological spacetimes and discuss resulting physical implications of quantum gravitational effects on the non-singular evolution of the LQC universe. The goal of this chapter is to not provide an exhaustive review of various techniques one may use and various questions one may pose to understand the nature of singularity resolution (One example is to pose the question with respect to unitary evolution and loss of determinism. See, e.g., [78].), but only to give a flavor of the physical implications resulting from underlying quantum geometry assuming the validity of effective spacetime description in LQC. For completeness and to make the material accessible to a broader audience, we also discuss some aspects of classical Hamiltonian cosmology in metric variables and the nature of singularities. The topics in each section are organized as follows. In section "Hamiltonian Cosmology", the Hamiltonian formulation of GR is reviewed in the metric as well as the Ashtekar-Barbero variables. The classical Hamiltonian constraints and the corresponding dynamical equations for the spatially flat FLRW universe and the Bianchi-I universe are also discussed briefly. In section "Nature of Classical Singularities: Types, Strength, and Shapes", the types and the strength of the classical singularities that can be encountered in the cosmological spacetimes are addressed. For the spacetimes with anisotropies, we also review different shapes of singularities that arise from the anisotropic behavior of the directional scale factors when a singularity is approached. In section "Loop Quantum Cosmology: Spatially Flat Isotropic Model", we go through in some detail the construction of the kinematic Hilbert space and the quantum Hamiltonian constraint operator in the spatially flat, homogeneous, and isotropic LQC universe. The properties of the resulting quantum difference equation are discussed along with the viability and the uniqueness of the improved dynamics (the $\overline{\mu}$ scheme). Section "The Effective Dynamics of Isotropic Model" is devoted to a summary of the effective dynamics for the $k = 0$ homogeneous and isotropic LQC universe and its phenomenological applications. The effective dynamics, whose validity is assumed in all regimes, is extensively used in showing the generic resolution of the strong singularities and investigating the extensions of the inflationary, and ekpyrotic- and matter-bounce scenarios in the LQC universe. In section "Loop Quantization in the Presence of Anisotropies and Inhomogeneities", we move onto the loop quantization of spacetimes with anisotropies and inhomogeneities. In particular, we summarize the novel features of the loop quantization of Bianchi-I and Gowdy models and some of their physical implications on the inflationary paradigm. In section "Beyond Standard LQC: Incorporating Additional Elements from LQG", we discuss different variants of LQC, such as the modified LQC models. Here emphasis is on their distinct physical predictions as compared with those from standard LQC. Finally, section "Summary and Outlook" is attributed to a summary of the chapter.

本章我们总结了圈量子宇宙学 (LQC) 在解决各类宇宙学时空中奇点问题方面的主要成果，并讨论了量子引力效应对 LQC 宇宙非奇异演化带来的物理启发。本章的目标并非全面综述可用于理解奇点 Resolution 本质的各类研究方法与各类问题 (例如从幺正演化与确定性丧失角度提出的相关问题，参见文献 [78])，而是仅在假定 LQC 有效时空描述成立的前提下，阐述底层量子几何带来的物理意义。为保证内容完整，同时方便更广泛的读者阅读，我们还会讨论度量变量下经典哈密顿宇宙学的相关内容与奇点的本质。各节主题安排如下: 在「哈密顿宇宙学」一节中，我们回顾了广义相对论 (GR) 在度量变量与 Ashtekar-Barbero 变量下的哈密顿表述，并简要讨论了空间平坦 FLRW 宇宙与 Bianchi-I 宇宙的经典哈密顿约束及对应动力学方程。在「经典奇点的本质: 类型、强度与形态」一节中，我们探讨了宇宙学时空中可能出现的经典奇点的类型与强度；针对各向异性时空，我们还回顾了趋近奇点时，方向标度因子的各向异性行为所导致的不同奇点形态。在「圈量子宇宙学: 空间平坦各向同性模型」一节中，我们详细介绍了空间平坦、均匀各向同性 LQC 宇宙中运动学希尔伯特空间与量子哈密顿约束算符的构造，讨论了所得量子差分方程的性质，以及改进动力学 (即 $\bar{\mu}$ 方案) 的合理性与唯一性。「各向同性模型的有效动力学」一节专门总结了 $k = 0$ 均匀各向同性 LQC 宇宙的有效动力学及其唯象学应用。在所有区域均假定有效的这套动力学被广泛用于证明强奇点的通有消解，并研究 LQC 宇宙中暴胀场景、循环暴胀场景与物质反弹场景的延拓。在「各向异性与非均匀性存在时的圈量子化」一节中，我们转向讨论含各向异性与非均匀性时空的圈量子化，特别总结了 Bianchi-I 模型与 Gowdy 模型圈量子化的新特征，及其对暴胀范式的若干物理启示。在「超越标准 LQC: 融入 LQG 的额外元素」一节中，我们讨论了 LQC 的不同变体，例如修正 LQC 模型，重点介绍了这类变体与标准 LQC 相比不同的物理预言。最后，「总结与展望」一节对全章内容做了总结。

Hamiltonian Cosmology

哈密顿宇宙学

In this section we provide an overview of the Hamiltonian formulation of general relativity (GR) which serves as the first step to understand the main procedures of loop quantization of cosmological spacetimes in loop quantum cosmology. We start with the Arnowitt-Deser-Misner (ADM) decomposition of the classical spacetime and present the Hamiltonian and the spatial and temporal diffeomorphism constraints first in terms of metric variables and then the Ashtekar-Barbero variables. We then specialize to the spatially flat, homogeneous, and isotropic spacetime to obtain the relevant Hamiltonian constraint which leads to the classical Friedmann and Raychaudhuri equations. Finally, we summarize the main results of the Hamiltonian formulation of anisotropic Bianchi-I spacetime and obtain the classical dynamical equations.

本节我们概述广义相对论 (GR) 的哈密顿表述，它是理解圈量子宇宙学中宇宙时空圈量子化核心步骤的基础。我们从经典时空的阿诺维特-德瑟-米斯纳 (ADM) 分解出发，先后以度规变量和阿西特卡-巴贝罗变量给出哈密顿量，以及空间微分同胚约束与时间微分同胚约束。随后我们特例化到空间平坦的均匀各向同性时空，推导出对应的哈密顿约束，由此得到经典弗里德曼方程和瑞查得符里方程。最后，我们总结各向异性比安基 I 型时空哈密顿表述的核心结果，并推导出经典动力学方程。

The Hamiltonian Formulation of General Relativity in ADM Decomposition and Ashtekar-Barbero Variables

ADM 分解下广义相对论的哈密顿表述与 Ashtekar-Barbero 变量

The 3+1 foliation of globally hyperbolic spacetime was first applied to GR by Arnowitt, Deser and Misner in their seminal work [79]. In this formalism, a 4-dimensional manifold $\mathcal{M}$ with a Lorentzian metric g can be decomposed into $\mathcal{M} := \sum \times \mathbb{R}$, where $\sum$ is the space-like hypersurface and $\mathbb{R}$ denotes the real line. Correspondingly, the 4-metric of the manifold $g_{\mu\nu}$ with μ, ν running from 0 to 4 can be expressed in terms of the lapse function N , the shift vector N^a , and the spatial metric q_{ab} , where $a, b = 1, 2, 3$. To be specific, different components of the metric can be expressed as

Arnowitt、Deser 和 Misner 在其开创性工作 [79] 中首次将整体双曲时空的 3+1 叶分解应用于广义相对论。在该形式体系中，带有洛伦兹度规 g 的 4 维流形 $\mathcal{M}$ 可分解为 $\mathcal{M} := \sum \times \mathbb{R}$，其中 $\sum$ 为类空超曲面，$\mathbb{R}$ 为实线。相应地，指标 μ, ν 取值从 0 到 3 的流形 $g_{\mu\nu}$ 的 4 维度规可由移轴函数 N 、位移矢量 N^a 和空间度规 q_{ab} 表示，其中 $a, b = 1, 2, 3$ 。具体来说，度规的各分量可写为

$$g_{00} = -N^2 + N_a N^a, \; g_{0a} = N_a, \; g_{ab} = q_{ab}. \tag{1}$$

Using the above decomposition of the 4-metric in the Einstein-Hilbert action of GR, one can obtain the canonical form of the action in the phase space spanned by the spatial metric and its conjugate momentum π^{ab} . This turns out to be [79]

将上述 4 维度规分解代入广义相对论的爱因斯坦-希尔伯特作用量，即可得到由空间度规及其共轭动量 π^{ab} 张成的相空间中作用量的正则形式，结果为 [79]

$$S = \int \mathrm{d}^4 x \left(\pi^{ab}\dot{q}_{ab} - N\mathcal{H} - N^a \mathcal{H}_a\right). \tag{2}$$

Here the conjugate momentum π^{ab} carries the information of the rate of change of the spatial metric and is given by

此处共轭动量 π^{ab} 包含了空间度量变化率的信息，其表达式为

$$\pi^{ab} = \sqrt{q}\left(K^{ab} - K q^{ab}\right). \tag{3}$$

Here K^{ab} stands for the extrinsic curvature of the hypersurface defined by $K_{ab} = \left(-\dot{q}_{ab} + D_a N_b + D_b N_a\right)/2N$ and q denotes the determinant of the spatial metric. D_a is the covariant derivative with respect to the spatial metric. The Hamiltonian constraint (or the scalar or the energy constraint) and the spatial diffeomorphism (or the momentum constraint) turn out to be, respectively, as [79]

其中 K^{ab} 是由 $K_{ab} = \left(-\dot{q}_{ab} + D_a N_b + D_b N_a\right)/2N$ 定义的超曲外曲率，q 为空间度规的行列式，D_a 是对应空间度规的协变导数。哈密顿约束 (又称标量约束或能量约束) 与空间微分同胚约束 (又称动量约束) 分别为 [79]

$$\mathcal{H} = \frac{2\kappa}{\sqrt{q}}\left(\pi^{ab}\pi_{ab} - \frac{\pi^2}{2}\right) - \frac{\sqrt{q}}{2\kappa}{}^{(3)}R \approx 0, \tag{4}$$

$$\mathcal{H}_a = -2\partial_c\left(\gamma_{ab}\pi^{bc}\right) + \pi^{bc}\partial_a\gamma_{bc} \approx 0, \tag{5}$$

where $\kappa = 8\pi G$ and ${}^{(3)}R$ denotes the intrinsic curvature of the 3-dimensional spatial hypersurface. Since the lapse function and the shift vector in the action (2) act as the Lagrange multipliers, the Hamiltonian and the diffeomorphism constraints weakly vanish on the trajectories of the physical solutions as indicated by ”⍰ 0” in Eqs. (4)-(5). These constraints are the first class constraints of the system, and their Poisson brackets constitute the fundamental constraint algebra in GR. Recall that GR is a fully constrained system since the total Hamiltonian $H = \int \mathrm{d}^3x\,(N\mathcal{H} + N^a\mathcal{H}_a)$ weakly vanishes on the dynamical trajectories of the physical solutions. As a result, the evolution of all the physical observables which are supposed to commute with all the first class constraints is frozen, leading to the problem of time. To extract physics from this frozen formalism, we note that only those quantities are physically observable which are invariant under spacetime diffeomorphisms, such as Dirac observables which commute with the Hamiltonian. To construct relevant Dirac observables which help extract dynamics, we introduce appropriate reference fields or clock degrees of freedom in the relational formalism [80-86]. In the canonical formalism these reference fields have been studied in various contexts. In applications using framework of LQG, examples include the reduced phase quantization which is based on quantizing phase space of gauge-invariant quantities [87-90], cosmological perturbations in the presence of reference fields [91-95], and singularity resolution in LQC [9,96].

> 其中 $\kappa = 8\pi G$ 和 ${}^{(3)}R$ 表示 3 维空间超曲面的内禀曲率。由于作用量 (2) 中的移轴函数和位移矢量充当拉格朗日乘子，哈密顿约束与微分同胚约束在物理解的轨迹上弱为零，即式 (4)-(5) 中的”⍰ 0”。这些约束是系统的第一类约束，它们的泊松括号构成了广义相对论的基本约束代数。我们知道广义相对论是完全约束系统，因为总哈密顿量 $H = \int \mathrm{d}^3x\,(N\mathcal{H} + N^a\mathcal{H}_a)$ 在物理解的动力学轨迹上弱为零。因此，所有与所有第一类约束对易的物理可观测量的演化都是冻结的，这就产生了时间问题。为了从这种冻结形式中提取物理，我们注意到只有在时空微分同胚下不变的量才是物理可观测量，例如与哈密顿量对易的狄拉克可观测量。为了构造能够提取动力学的相关狄拉克可观测量，我们在关系形式论 [80-86] 中引入合适的参考场或时钟自由度。在正则形式论中，这些参考场已在多种背景下得到研究。在 loop 量子引力 (LQG) 框架的应用中，相关研究包括基于规范不变量相空间量子化的约化相位量子化 [87-90]、存在参考场时的宇宙学扰动 [91-95]，以及圈量子宇宙学 (LQC) 中的奇点解决 [9,96]。

Following Dirac’s approach for quantization of constrained systems [97], and using the ADM formulation [98], DeWitt obtained ”Einstein-Schrödinger equation” [11] which was later known as the Wheeler-DeWitt equation. In the case of finite degrees of freedom, its implications were first studied in the mini-superspace setting by Misner [99]. However, the convoluted form of the Hamiltonian which is non-polynomial in the metric variables makes it very difficult to obtain the physical solutions from the Wheeler-Dewitt equation in a general setting. Even in the mini-superspace setting of symmetry reduced models, such as a homogeneous and isotropic cosmological spacetime, where the relevant Wheeler-Dewitt equation can be exactly solved, physical solutions encounter singularities. An example is the case of the spatially flat homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime filled with a massless scalar field [9]. In fact, the consistent quantum probability of singularity to occur in this model is unity including for wavefunctions which consist of arbitrary superpositions of expanding and contracting branches [19,20]. While it is possible to fine-tune the avoidance of singularity in the WDW theory in some cases, such as in the presence of suitable potentials or exotic matter, such a resolution is often problematic and non-generic. This reveals one of the fundamental limitations of the WDW theory which not being based on a quantum theory of geometry fails to generically resolve the singularity. As we discuss later, this situation changes dramatically when using Ashtekar-Barbero variables in LQC.

遵循狄拉克约束系统量子化方法 [97]，并利用 ADM 表述 [98]，德威特得到了后来被称为惠勒-德威特方程的“爱因斯坦-薛定谔方程”[11]。对于有限自由度的情况，米斯纳首先在微超空间框架下研究了它的含义 [99]。然而，哈密顿量形式复杂，且在度规变量中为非多项式，使得在一般背景下从惠勒-德威特方程得到物理解非常困难。即使在对称约化模型的微超空间框架下，例如均匀各向同性宇宙学时空中，相关惠勒-德威特方程可以精确求解，物理解仍然会遇到奇点。填充无质量标量场的空间平坦均匀各向同性弗里德曼-勒梅特-罗伯逊-沃尔克 (FLRW) 时空就是一个例子 [9]。事实上，在这个模型中，无论波函数是膨胀分支和收缩分支的任意叠加，奇点发生的一致量子概率都是 1[19,20]。虽然在 WDW 理论中可以通过精细调谐在某些情况下 (例如存在合适势场或奇异物质时) 避免奇点，但这种解决方式通常存在问题，不具有普遍性。这暴露了 WDW 理论的一个根本局限性：它不基于量子几何理论，因此无法普遍解决奇点问题。正如我们后文将要讨论的，在圈量子宇宙学中使用阿西特卡-巴贝罗变量时，这种情况会发生巨大改变。

In terms of the Ashtekar-Barbero variables-the connection A_a^i and the densi-tized triad E_i^a [100], GR can be reformulated as a gauge field theory at least at a kinematical level using the internal SU(2) symmetry of variables. They are defined explicitly by

借助阿西特卡-巴贝罗变量——联络 A_a^i 和密化标架 E_i^a [100]，至少在运动学层面，可以利用变量的内部 SU(2) 对称性将广义相对论重新表述为规范场论。这些变量的显式定义如下:

$$E_i^a = \sqrt{|q|}e_i^a,\ A_a^i = \Gamma_a^i + \gamma K_a^i, \tag{6}$$

where i, j are the internal SU(2) indices running from 1 to 3 and γ is the Barbero-Immirzi parameter, e_i^a is the triad, $K_a^i\ (= K_{ab}e^{bi}\ )$ is the extrinsic curvature with one of its indices projected onto the internal frame, and Γ_a^i is the spin connection satisfying the condition that it be compatible with the triad, namely $D_a e_b^i = 0$, resulting in $\Gamma_a^i = -\frac{1}{2}\varepsilon^{ijk}e_j^b\left(\partial_a e_b^k - \partial_b e_a^k + \delta_{mn}e_k^l e_a^m \partial_l e_b^n\right)$ with ε^{ijk} being the Levi-

其中 i, j 是取值 1 到 3 的内部 SU(2) 指标，γ 是巴贝罗-伊米尔齐参数，e_i^a 是标架，$K_a^i\ (= K_{ab}e^{bi}\ )$ 是外曲率，其一个指标投影到内部标架，Γ_a^i 是满足与标架相容条件的自旋联络，即 $D_a e_b^i = 0$ ，由此得到 $\Gamma_a^i = -\frac{1}{2}\varepsilon^{ijk}e_j^b\left(\partial_a e_b^k - \partial_b e_a^k + \delta_{mn}e_k^l e_a^m \partial_l e_b^n\right)$ ，其中 ε^{ijk} 是列维-

Civita symbol. In terms of the new variables, the constraints of GR can be rewritten as [6, 101]

奇维塔符号。使用新变量时，广义相对论的约束可以改写为 [6, 101]

$$G_i = \partial_a E_i^a + \varepsilon_{ijk}A_a^j E^{ak} \approx 0,\ \mathcal{H}_a = \frac{1}{\gamma}F_{ab}^i E_i^b - \frac{1+\gamma^2}{\gamma}K_a^i G_i \approx 0, \tag{7}$$

and

和

$$\mathcal{H} = \frac{1}{\sqrt{|q|}} E^a_j E^b_k \left(\varepsilon_i^{jk} F^i_{ab} - \left(1+\gamma^2\right) \left(K^j_a K^k_b - K^j_b K^k_a \right) \right)$$

$$+ \frac{1+\gamma^2}{\gamma} G^i \partial_a \frac{E^a_i}{\sqrt{|q|}} \approx 0, \tag{8}$$

where the field strength of connection A^i_a is given by $F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \varepsilon^i_{jk} A^j_a A^k_b$. The first constraint in Eq. (7) is the Gauss constraint which generates the internal SU(2) symmetry. The Hamiltonian constraint given in Eq. (8) is a linear combination of a Euclidean term which is proportional to the field strength and a Lorentzian term proportional to the extrinsic curvature and the Gauss constraint. In the quantum theory, the resulting form of the Hamiltonian constraint depends on the associated quantization ambiguities in loop quantizing these terms. In the case of the homogeneous models, due to the underlying symmetry reduction, it is possible to combine the Euclidean and Lorentzian terms (From a general perspective, in full LQG, it is possible to decompose the Lorentzian part of the Hamiltonian constraint into the one proportional to the Euclidean term and the spatial Ricci scalar. In a spatially flat FLRW universe, with a vanishing spatial curvature, the Lorentzian part becomes a multiple of the Euclidean term.). Thus, a loop quantization based on utilizing this symmetry reduction is expected to be different from the one where quantization treats Euclidean and Lorentzian terms independently. The standard LQC is based on the former strategy, while the latter method results in modified versions of LQC. We would later see that different treatments of the Euclidean and Lorentzian parts result in different physical predictions than from the standard LQC.

其中联络的场强 A^i_a 由 $F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \varepsilon^i_{jk} A^j_a A^k_b$ 给出。式 (7) 中的第一约束是生成内部 SU(2) 对称性的高斯约束。式 (8) 给出的哈密顿约束是欧几里得项与洛伦兹项的线性组合，前者正比于场强，后者正比于外曲率与高斯约束。在量子理论中，哈密顿约束的最终形式取决于对这些项进行圈量子化时相关的量子化不确定性。在齐次模型的情况下，由于基础对称性约化，我们可以将欧几里得项与洛伦兹项结合起来 (从一般角度来看，在完整圈量子引力 (LQG) 中，可以将哈密顿约束的洛伦兹部分分解为正比于欧几里得项的部分和空间里奇标量部分；在空间平坦、空间曲率为零的 FLRW 宇宙中，洛伦兹部分成为欧几里得项的倍数)。因此，利用这种对称性约化得到的圈量子化预期会不同于将欧几里得项与洛伦兹项分开量子化的结果。标准圈量子宇宙学 (LQC) 采用前一种策略，后一种方法则得到修正版本的 LQC。我们在后文会看到，对欧几里得部分与洛伦兹部分的不同处理，会得到与标准 LQC 不同的物理预言。

Spatially Flat, Homogeneous, and Isotropic Spacetime: Classical Aspects

空间平坦、均匀各向同性时空: 经典方面

Given that most of the analysis in LQC so far has focused on understanding implications for the spatially flat, homogeneous, isotropic model, in the following we provide a brief overview of the Hamiltonian aspects in the classical theory for this model. The spacetime metric in the spatially flat FLRW universe is given by

鉴于目前圈量子宇宙学 (LQC) 的大部分分析都聚焦于探究空间平坦、均匀各向同性模型的相关推论，下文我们将简要概述该模型经典理论中的哈密顿方面内容。空间平坦 FLRW 宇宙的时空度规由下式给出

$$\mathrm{d}s^2 = -N^2\,\mathrm{d}t^2 + a^2(t)\,\delta_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b, \tag{9}$$

where $a(t)$ denotes the scale factor of the universe. Its spatial topology can be either a 3-torus $\mathbb{T}^3$ or non-compact $\mathbb{R}^3$. For the torus $\mathbb{T}^3$, the co-moving length of each spatial direction is naturally confined within the range $x_a \in (0, l_o)$ which makes the spatial integrals in the Hamiltonian finite. For the $\mathbb{R}^3$ topology, a fiducial cell $\mathcal{V}$ with co-moving volume V_o is required. All the spatial integrals are restricted to the fiducial cell so that no divergences would be encountered. In this way, there is also a well-defined symplectic structure even with $\mathbb{R}^3$ spatial topology. In the following analysis, we take the $\mathbb{R}^3$ spatial topology. In addition to the gravitational degrees of freedom, for the matter content, we consider the simplest case of a massless scalar field ϕ which serves as a reference field for relational dynamics. Thus, the mini-superspace of the spatially flat FLRW universe consists of the scale factor a and its conjugate momentum π_a as well as the canonical pair ϕ and π_ϕ . In terms of these two canonical pairs, the action in (2) with additional contributions from the matter sector reduces to

其中 $a(t)$ 表示宇宙的标度因子。其空间拓扑既可以是三维环面 $\mathbb{T}^3$ ，也可以是非紧致的 $\mathbb{R}^3$ 。对于环面 $\mathbb{T}^3$ ，每个空间方向的共动长度自然被限制在范围 $x_a \in (0, l_o)$ 内，这使得哈密顿中的空间积分是有限的。对于 $\mathbb{R}^3$ 拓扑，则需要引入一个基准格胞 $\mathcal{V}$ ，其共动体积为 V_o 。所有空间积分都限制在基准格胞内，因此不会出现发散。通过这种处理，即使是 $\mathbb{R}^3$ 空间拓扑也能得到定义良好的辛结构。在下文的分析中，我们采用 $\mathbb{R}^3$ 空间拓扑。除引力自由度外，对于物质组分，我们考虑无质量标量场 ϕ 的最简单情况，该标量场可作为关系动力学的参考场。因此，空间平坦 FLRW 宇宙的迷你超空间由标度因子 a 及其共轭动量 π_a ，再加上正则对 ϕ 和 π_ϕ 构成。用这两组正则对表示，(2) 式中包含物质部分额外贡献的作用量可约化为

$$S = \int \mathrm{d}t \left\{ V_o \left(\pi_\phi \dot{\phi} + \pi_a \dot{a} \right) - N\mathcal{H} \right\}, \tag{10}$$

which implies the standard Poisson brackets $\{a, \pi_a\} = \{\phi, \pi_\phi\} = 1/V_o$. Note that due to the homogeneity of the background spacetime, the spatial diffeomorphism constraint identically vanishes, while the background Hamiltonian constraint reduces to

它给出标准泊松括号 $\{a, \pi_a\} = \{\phi, \pi_\phi\} = 1/V_o$ 。注意，由于背景时空具有均匀性，空间微分同胚约束恒为零，而背景哈密顿约束可约化为

$$\mathcal{H} = V_o \left(-\frac{\kappa \pi_a^2}{12a} + \frac{\pi_\phi^2}{2a^3} \right). \tag{11}$$

The corresponding Hamilton's equations can be obtained by evaluating the Poisson bracket between the canonical variables and the background Hamiltonian constraint. To be specific, one finds

我们可以通过计算正则变量与背景哈密顿约束的泊松括号得到对应的哈密顿方程，具体结果如下:

$$\dot{a} = \{a, \mathcal{H}\} = -\frac{\kappa \pi_a}{6a}, \ \dot{\pi}_a = \frac{3\pi_\phi^2}{2a^4} - \frac{\kappa \pi_a^2}{12a^2}. \tag{12}$$

It is straightforward to obtain the Friedmann and Raychaudhuri equations from above equations. The classical Friedmann equation can be obtained by squaring the Hamilton’ s equation of the scale factor and then expressing π_a^2 in terms of the energy density of the scalar field resulting from the vanishing of the background Hamiltonian constraint. On the other hand, the Raychaudhuri equation can be derived from the Hamilton's equations of the scale factor and its conjugate momentum. Collectively, the classical Friedmann and Raychaudhuri equations take the form

从上述方程很容易推导出弗里德曼方程和瑞查得符里方程。将标度因子的哈密顿方程平方后，利用背景哈密顿约束为零的条件，把 π_a^2 用标量场的能量密度表示，即可得到经典弗里德曼方程。而瑞查得符里方程可由标度因子及其共轭动量的哈密顿方程推导得到。经典弗里德曼方程和瑞查得符里方程合写为如下形式:

$$H^2 = \frac{8\pi G}{3}\rho, \ \frac{\ddot{a}}{a} = -4\pi G(\rho + 3P), \tag{13}$$

where the Hubble rate is defined via $H = \dot{a}/a$ and for a massless scalar field $P = \rho = \dot{\phi}^2/2$. Although both of the action and the background Hamiltonian contain the fiducial volume V_o , the resulting equations of motion are independent of V_o , implying that the classical dynamics remains unaffected by any other choice of the fiducial cell. Using the Hamilton's equations for the matter sector, it is also easy to check that the scalar field satisfies the Klein-Gordon equation or equivalently the matter-energy conservation law:

其中哈勃率由 $H = \dot{a}/a$ 定义，且对于无质量标量场满足 $P = \rho = \dot{\phi}^2/2$ 。尽管作用量和背景哈密顿都包含基准体积 V_o ，但最终得到的运动方程与 V_o 无关，这说明选择其他基准格胞不会改变经典动力学。利用物质部分的哈密顿方程，也很容易验证标量场满足克莱因-戈登方程，等价于物质能量守恒定律:

$$\dot{\rho} + 3H(\rho + P) = 0. \tag{14}$$

We can easily see these dynamical equations result in a singularity for a given choice of matter. For matter which has a fixed equation of state $w = P/\rho$, the conservation law results in $\rho \propto a^{-3(1+w)}$. For matter satisfying weak energy condition, $\rho \geq 0$ and $\rho + P \geq 0$, as the scale factor approaches zero, the energy density and pressure diverge. This corresponds to the big-bang singularity in the backward evolution (or the big crunch singularity in the forward evolution) where the Friedmann and Raychaudhuri equations break down.

我们可以很容易看出，对于给定的物质选择，这些动力学方程会导致奇点的存在。对于物态方程固定为 $w = P/\rho$ 的物质，守恒定律给出 $\rho \propto a^{-3(1+w)}$ 。对于满足弱能量条件的物质，有 $\rho \geq 0$ 和 $\rho + P \geq 0$ ，当标度因子趋近于零时，能量密度和压强都会发散。这对应于反向演化中的大爆炸奇点 (或正向演化中的大挤压奇点)，此时弗里德曼方程和瑞查得符里方程不再成立。

Let us now discuss how to obtain these equations using the Ashtekar-Barbero variables. Owing to the homogeneity of the spacetime, the connection A_a^i and the triad E_i^a can be expressed in terms of their isotropic counterparts c and p as [12]

下面我们讨论如何利用阿西特卡-巴贝罗变量得到这些方程。由于时空具有均匀性，联络 A_a^i 和三元组 E_i^a 可以通过对应的各向同性形式 c 和 p 表示为 [12]

$$A_a^i = cV_o^{-1/3}\mathring{\omega}_a^i, \ E_i^a = pV_o^{-2/3}\sqrt{\mathring{q}}\mathring{e}_i^a. \tag{15}$$

Here $\mathring{e}_i^{\ a}$ and $\mathring{\omega}_a^{\ i}$ are the fiducial triads and co-triads compatible with the fiducial metric which is simply $\mathring{q}_{ab} = \delta_{ab}$ in the spatially flat FLRW spacetime. The symmetry reduced isotropic connection and triad satisfy the Poisson bracket $\{c, p\} = 8\pi G\gamma/3$. The triad is kinematically related to the metric variables as $|p| = V_o^{2/3}a^2$ where the modulus sign over the triad arises to account two possible orientations. The relationship between the connection and the scale factor is a dynamical one to be obtained from the Hamilton's equation for the triad. In the classical theory it is given by $c = \gamma V_o^{1/3}\dot{a}$. It is important to note that the latter relation changes in LQC.

此处 $\mathring{e}_i^{\ a}$ 和 $\mathring{\omega}_a^{\ i}$ 是与基准度规兼容的基准三元组和基准余三元组，在空间平坦的 FLRW 时空中，基准度规就是 $\mathring{q}_{ab} = \delta_{ab}$ 。经对称性约化得到的各向同性联络和三元组满足泊松括号 $\{c, p\} = 8\pi G\gamma/3$ 。三元组与度规变量运动学关系为 $|p| = V_o^{2/3}a^2$ ，对三元组取绝对值是为了兼顾两种可能的取向。联络和尺度因子之间的关系是动力学关系，需要从三元组的哈密顿方程得到，经典理论中该关系为 $c = \gamma V_o^{1/3}\dot{a}$ 。需要注意的是，在圈量子宇宙学中这一关系会发生改变。

As discussed later, in improved dynamics of LQC [10], it is more convenient to work with another set of phase space variables related to (c, p) variables as [15]

稍后我们会讨论，在圈量子宇宙学的改进动力学中 [10]，使用另一组与 (c, p) 变量相关的相空间变量会更方便，关系为 [15]

$$b \ := \ c/|p|^{1/2}, \ v \ := \ \mathrm{sgn}\,(p)\,|p|^{\frac{3}{2}}, \tag{16}$$

where $\mathrm{sgn}\,(p) = \pm 1$ for the same/opposite orientation of the physical and fiducial triads. These variables satisfy the Poisson bracket $\{b, v\} = 4\pi G\gamma$. In terms of the new variables b and v , the background Hamiltonian for the spatially flat FLRW universe can be written in the form

其中 $\mathrm{sgn}\,(p) = \pm 1$ 对应物理三元组与基准三元组同向/反向取向。这些变量满足泊松括号 $\{b, v\} = 4\pi G\gamma$ 。用新变量 b 和 v ，空间平坦 FLRW 宇宙的背景哈密顿量可以写为如下形式

$$\mathcal{H} = -\frac{3b^2\,|v|}{8\pi G\gamma^2} + \frac{p_\phi^2}{2\,|v|}, \tag{17}$$

with $p_\phi = V_o\pi_\phi$. It is straightforward to derive the Hamilton’s equations for v and b which result in the Friedmann and Raychaudhuri equations using the vanishing of the background Hamiltonian constraint. From the Hamilton’s equations of v and ϕ , it is straightforward to obtain the relation

其中 $p_\phi = V_o\pi_\phi$ 。利用背景哈密顿约束等于零，很容易推导出 v 和 b 的哈密顿方程，最终得到弗里德曼方程和瑞查得符里方程。通过 v 和 ϕ 的哈密顿方程，可以很容易得到关系

$$\frac{\mathrm{d}v}{\mathrm{d}\phi} = \frac{3b|v|^2}{p_\phi\gamma} \tag{18}$$

Let us note that in the classical theory b is related with the Hubble rate via $b = \gamma H$, and then using the classical Friedmann equation, the above differential equation turns out to be equivalent to $\mathrm{d}\phi = \frac{\mathrm{sgn}(p_\phi)}{\sqrt{12\pi G|v|}}\mathrm{d}v$, leading to the generic solution

我们注意到，在经典理论中 b 与哈勃率通过 $b=\gamma H$ 关联，再利用经典弗里德曼方程，可以推导出上述微分方程等价于 $\mathrm{d}\phi=\frac{\mathrm{sgn}(p_\phi)}{\sqrt{12\pi G|v|}}\mathrm{d}v$，由此得到通解

$$\phi=\pm\frac{1}{\sqrt{12\pi G}}\ln\left|\frac{v}{v_0}\right|+\phi_0, \tag{19}$$

where v_0 and ϕ_0 are integration constants and $\pm$ sign is determined by the sign of the constant momentum. The above solutions represent two disjoint trajectories which describe an expanding and a contracting universe, respectively. The singularity is inevitable when the volume reaches zero in the past of an expanding universe which thus emerges from a big-bang singularity or in the future of a contracting universe that ends up with a big crunch singularity. A plot of these solutions is shown in Fig. 1 as dashed curves.

其中 v_0 和 ϕ_0 是积分常数，$\pm$ 的符号由常数动量的符号决定。上述解对应两条不相交的轨迹，分别描述膨胀宇宙和收缩宇宙。对于膨胀宇宙，过去演化中体积会趋近于零，对于收缩宇宙，未来演化中体积会趋近于零，因此奇点不可避免: 膨胀宇宙从大爆炸奇点诞生，收缩宇宙最终坍缩成大挤压奇点。这些解在图 1 中用虚线绘出。

The Hamiltonian Formulation of Anisotropic Spacetime: Bianchi-I Model

各向异性时空的哈密顿表述:Bianchi-I 模型

The Bianchi-I spacetime is one of the simplest models in which the effects of anisotropies on the physics near the classical singularities can be systematically investigated. It has vanishing intrinsic curvature, and its isotropic limit is the spatially flat FLRW universe. Unlike in the isotropic models where the big-bang singularity is characterized by a vanishing scale factor of the universe, the structure of the singularities in the anisotropic spacetimes becomes much richer since the directional scale factors can evolve in different ways when the singularity is approached. As discussed later in the next section, the shape of the singularity can be of a cigar, a pancake, a barrel, or a point depending on the way directional scale factors approach zero. As one of the simplest anisotropic models to study, the dynamics of Bianchi-I spacetime is important to understand as it plays a vital role in understanding the approach to singularities in the classical Bianchi-IX spacetime and in particular the Mixmaster dynamics [102].

Bianchi-I 时空是最简单的模型之一，可以系统研究经典奇点附近物理中的各向异性效应。它的内禀曲率为零，其各向同性极限是空间平坦的 FLRW 宇宙。各向同性模型中，大爆炸奇点以宇宙尺度因子趋近于零为特征，与之不同，各向异性时空的奇点结构丰富得多，因为在趋近奇点时，不同方向的尺度因子可以按不同方式演化。正如我们在下一节将会讨论的，根据各方向尺度因子趋近于零的方式，奇点的形状可以是雪茄形、煎饼形、桶形或点奇点。作为最简单的可研究各向异性模型之一，理解 Bianchi-I 时空的动力学十分重要，它对理解经典 Bianchi-IX 时空趋近奇点的过程，尤其是 Mixmaster 动力学，起到关键作用 [102]。

In the following, we consider the simplest case of a homogeneous Bianchi-I spacetime with a manifold $\mathcal{M}:=\sum\times\mathbb{R}$, where $\sum$ is topologically flat, and its metric is given by

下面我们考虑齐次 Bianchi-I 时空的最简单情况，其流形为 $\mathcal{M} := \sum \times \mathbb{R}$，其中 $\sum$ 拓扑平坦，它的度规由下式给出:

$$ds^2 = -N^2\, dt^2 + a_1^2(t)\, dx^2 + a_2^2(t)\, dy^2 + a_3^2(t)\, dz^2, \tag{20}$$

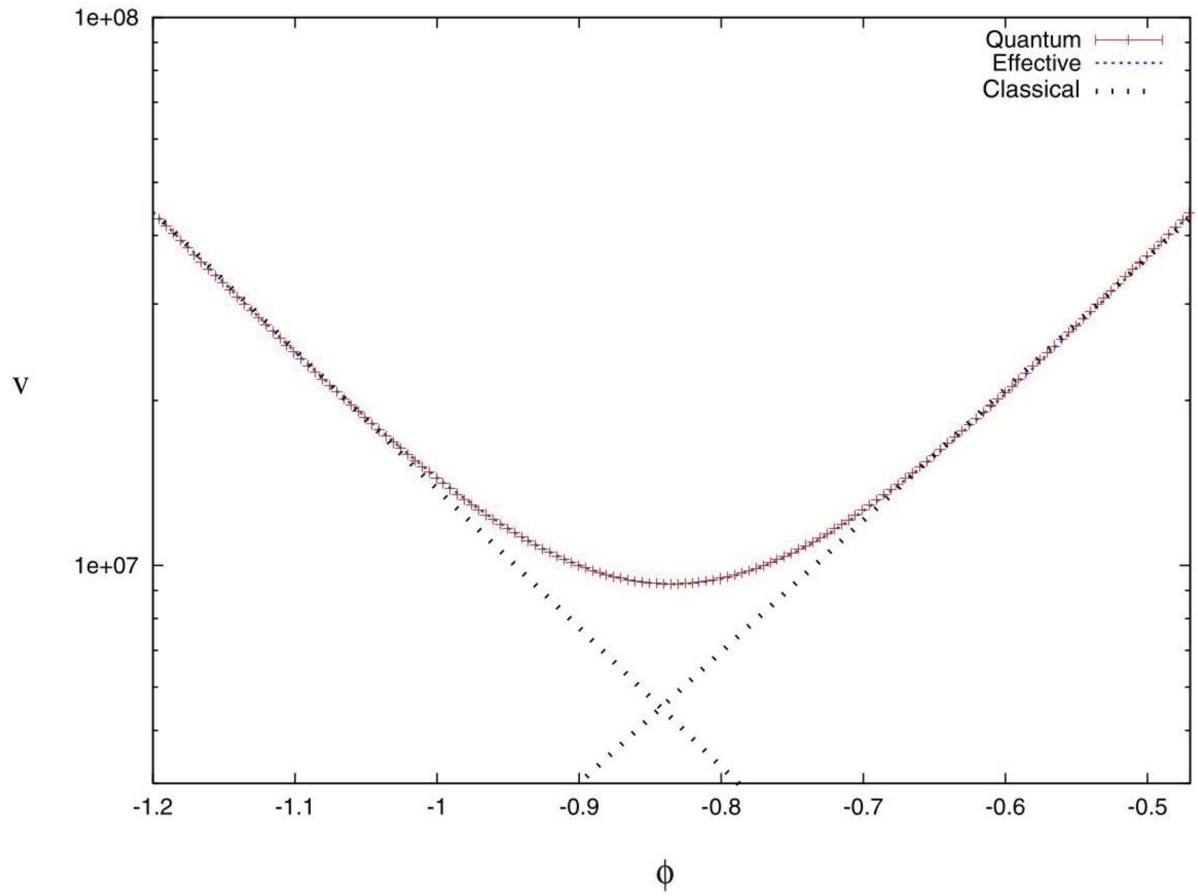

Fig. 1 In this figure, we show the trajectory of the expectation value of the volume operator from the numerical simulations of the quantum difference operator (42) which illustrates a non-singular evolution of the universe across a quantum bounce. The trajectories from the effective dynamics of LQC and the classical GR are depicted as well in order to make a comparison with the quantum evolution

图 1 本图展示了量子差分算子 (42) 数值模拟得到的体积算符期待值的轨迹，说明了宇宙穿越量子反弹的非奇异演化。同时为了和量子演化对比，也画出了 LQC 有效动力学和经典广义相对论的轨迹。

where a_i with $i = 1, 2, 3$ are the directional scale factors. Similar to the case of the spatially flat FLRW spacetime, in order to have a well-defined symplectic structure, a fiducial cell with l_i denoting its coordinate length along each side needs to be introduced. Therefore, the fiducial volume equals $V_o = l_1 l_2 l_3$. Due to the homogeneity of the spacetime, the Ashtekar-Barbero variables, namely the connection A_a^i and the densitized triad E_i^a , can be expressed in terms of c_i and p_i along each direction as [59]

其中带 $i = 1, 2, 3$ 的 a_i 是各方向的尺度因子。和空间平坦 FLRW 时空的情况类似，为了得到定义良好的辛结构，需要引入一个参考胞元，用 l_i 表示其各边的坐标长度。因此，参考体积等于 $V_o = l_1 l_2 l_3$ 。由于时空是齐次的，阿西特卡-巴贝罗变量，也就是联络 A_a^i 和密化三元 E_i^a ，可以按每个方向用 c_i 和 p_i 表示为 [59]:

$$A_a^i = \frac{c_i}{l_i}\mathring{\omega}_a^i,\ E_i^a = p_i l_i V_o^{-1}\sqrt{\mathring{q}}\mathring{e}_i^a. \tag{21}$$

Note there is no summation over index "i" in the above definitions. These canonical phase space pairs satisfy $\{c_i, p_j\} = 8\pi G\gamma\delta_{ij}$. Moreover, the triads are related to the metric variables by

注意上述定义中不对指标"i" 求和。这些正则相空间对满足 $\{c_i, p_j\} = 8\pi G\gamma\delta_{ij}$ 。此外，三元与度规变量满足以下关系:

$$p_1 = \varepsilon_1 l_2 l_3 a_2 a_3,\ p_2 = \varepsilon_2 l_1 l_3 a_1 a_3,\ p_3 = \varepsilon_3 l_1 l_2 a_1 a_2, \tag{22}$$

with $\varepsilon_i = \pm 1$ determined by the triad orientation. Since we are not considering fermions in the matter content, without any loss of generality, we choose the plus sign for the triads. The relationship between the connection and the metric variables is determined from the Hamilton's equations. Due to the homogeneity of the spacetime, the only nonvanishing constraint is the Hamiltonian constraint which in terms of the connection and triad takes the form [59]

其中 $\varepsilon_i = \pm 1$ 由三元的方向决定。由于我们不考虑物质组分中的费米子，不失一般性，我们为三元选取正号。联络和度规变量的关系由哈密顿方程决定。由于时空是齐次的，唯一非零的约束是哈密顿约束，用联络和三元表示的形式为 [59]:

$$\mathcal{H} = -\frac{N}{8\pi G\gamma^2 v}\left(c_1 p_1 c_2 p_2 + c_1 p_1 c_3 p_3 + c_2 p_2 c_3 p_3\right) + N v\rho, \tag{23}$$

where $v = V_o a_1 a_2 a_3$ denotes the physical volume of the fiducial cell and ρ is the matter-energy density. From the Hamilton's equations of the triads, one finds

其中 $v = V_o a_1 a_2 a_3$ 表示参考胞元的物理体积，ρ 是物质能量密度。从三元的哈密顿方程可以得到:

$$c_i = \gamma l_i \dot{a}_i, \tag{24}$$

where the lapse is already set to unity so that the time derivative is with respect to the cosmic time. Using the above relation in the Hamiltonian constraint, it is easily seen that the energy density of the matter sector is related with the directional Hubble rate via

这里已经将时移函数设为 1，因此时间导数是对宇宙时间求导。将上述关系代入哈密顿约束，不难看出物质部分的能量密度和各方向哈勃率满足:

$$\kappa\rho = H_1 H_2 + H_2 H_3 + H_3 H_1. \tag{25}$$

Now defining the mean Hubble rate $H = (H_1 + H_2 + H_3)/3$, we can obtain a generalized Friedmann equation in terms of the mean Hubble rate which takes the form

现在定义平均哈勃率 $H = (H_1 + H_2 + H_3)/3$ ，我们可以得到用平均哈勃率表示的广义弗里德曼方程，形式为:

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{\kappa}{3}\rho + \frac{\sum^2}{a^6}, \tag{26}$$

where the mean scale factor is defined by $a = (a_1 a_2 a_3)^{1/3}$ and $\sum^2$ is related with the shear scalar σ^2 and directional Hubble rates H_i via

其中平均尺度因子由 $a = (a_1 a_2 a_3)^{1/3}$ 定义，且 $\sum^2$ 通过下式与剪切标量 σ^2 和方向哈勃率 H_i 相关联

$$\sum^2 = \frac{1}{6}\sigma^2 a^6 = \frac{a^6}{18}\left((H_1 - H_2)^2 + (H_2 - H_3)^2 + (H_3 - H_1)^2\right). \tag{27}$$

Note that the above relation is kinematical and holds as it is in LQC. For the matter content which has no anisotropic stress, such as a massless/massive scalar field which is minimally coupled to gravity, $\sum^2$ is a constant of motion [58]. As a result, the anisotropies behave like a perfect fluid with an effective equation of state $w = 1$, and the shear scalar diverges as $\sigma^2 \propto a^{-6}$, which is faster than any perfect fluid with an equation of state $w < 1$ when the singularity at vanishing mean scale factor is approached. In those cases, the anisotropies dominate near the classical singularities. Finally, for the matter content which is composed of a massive scalar field, with its matter Hamiltonian given by the familiar form $\mathcal{H}_m = p_\phi^2/2v + vU(\phi)$, from the Hamilton's equations of the scalar field and its momentum, one can obtain the standard Klein-Gordon equation

请注意，上述关系是运动学关系，在圈量子宇宙学 (LQC) 中形式保持不变。对于无各向异性应力的物质组分，例如最小耦合引力的无质量/有质量标量场，$\sum^2$ 是一个运动常数 [58]。因此各向异性的表现等效于物态方程为 $w = 1$ 的理想流体，当趋近平均尺度因子为零的奇点时，标量剪切按 $\sigma^2 \propto a^{-6}$ 发散，速度快于任何物态方程为 $w < 1$ 的理想流体。在这些情况下，各向异性在经典奇点附近占主导地位。最后，对于由有质量标量场构成的物质组分，其物质哈密顿量为常见形式 $\mathcal{H}_m = p_\phi^2/2v + vU(\phi)$ ，从标量场及其动量的哈密顿方程出发，可以推导出标准克莱因-戈登方程

$$\ddot{\phi} + 3H\dot{\phi} + U_{,\phi} = 0, \tag{28}$$

where H is the mean Hubble rate and $U_{,\phi}$ denotes the derivative of the potential with respect to the scalar field. An important implication from the Klein-Gordon equation is that the dynamics of the scalar field is determined by the averaged effects of the anisotropies, namely the mean Hubble rate and the scalar field itself cannot detect the fine structure of the anisotropies.

其中 H 是平均哈勃率，$U_{,\phi}$ 表示势对标量场的导数。克莱因-戈登方程的一个重要推论是: 标量场的动力学由各向异性的平均效应决定，也就是说平均哈勃率和标量场本身无法感知各向异性的精细结构。

Nature of Classical Singularities: Types, Strength, and Shapes

经典奇点的性质: 类型、强度与形状

So far we have discussed the Hamiltonian formulation of isotropic and Bianchi-I model using the metric as well as Ashtekar-Barbero variables and obtained the dynamical equations which signal existence of singularities. In this section, we overview some of the properties of the singularities encountered in this cosmological setting. According to singularity theorems of Penrose, Hawking and Geroch [1,2], singularities are the boundaries of the spacetime beyond which the null/timelike geodesics cannot be extended in a unique deterministic manner. However, an important and complimentary aspect of singularities is their physical nature in terms of their strength which in literature has played a useful role in gaining insights on the nature of shell crossing and naked singularities in GR. In the following, we will first briefly discuss different types and strength of the singularities in the isotropic spacetimes. In the second part of the section, taking Bianchi-I model as an example, we address the different shapes of the singularities in an anisotropic spacetime.

迄今为止，我们已经利用度规以及阿西特卡-巴贝罗变量讨论了各向同性模型与比安基 I 型模型的哈密顿表述，得到了表明奇点存在的动力学方程。在本节中，我们将概述在该宇宙学背景下遇到的奇点的部分性质。根据彭罗斯、霍金和格罗赫的奇点定理 [1,2]，奇点是时空的边界，类光/类时测地线无法在边界外以唯一确定的方式延拓。不过，奇点一个重要的补充性质是其按强度划分的物理本质，在文献中，这一性质对深入理解广义相对论中壳交叉和裸奇点的本质发挥了重要作用。下文我们首先会简要讨论各向同性时空中不同类型和强度的奇点，在本节的第二部分，我们将以比安基 I 型模型为例，说明各向异性时空中奇点的不同形状。

Types and Strength of the Singularities

奇点的类型与强度

In the classical cosmology, as long as the null energy condition ($\rho + P \geq 0$) is satisfied, the spatially flat isotropic FLRW universe would inevitably encounter the big-bang/crunch singularity when evolved backward in time. Not even the inflationary scenario can help evade the big-bang singularity as the Borde-Guth-Vilenkin theorem proves [103]. In addition to the big-bang/crunch singularity, there can be other types of cosmological singularities, depending on the behavior of the scale factor a , the energy density ρ , and the pressure P as the singularity is approached [104, 105]. The latter two observables determine the behavior of the Ricci scalar R in a spatially flat universe since

在经典宇宙学中，只要满足零能量条件 ($\rho + P \geq 0$)，空间平直的各向同性 FLRW 宇宙沿时间反向演化必然会遭遇大爆炸/大挤压奇点。根据博尔德-古斯-维林金定理，即便是暴胀场景也无法规避大爆炸奇点 [103]。除大爆炸/大挤压奇点外，当接近奇点时，根据标度因子 a 、能量密度 ρ 和压强 P 的行为不同，还存在其他类型的宇宙学奇点 [104, 105]。在空间平直宇宙中，后两个可观测量决定了里奇标量 R 的行为，因为

$$R = 8\pi G\,(\rho - 3P)\,. \tag{29}$$

In general, for the matter with a non-dissipative equation of state $P = P(\rho)$, all the known cosmological singularities can be classified into the following categories [106]:

一般而言，对于具有非耗散物态方程 $P = P(\rho)$ 的物质，所有已知的宇宙学奇点可分为以下几类 [106]:

- Big-Bang/Crunch singularity: This type of singularities is characterized by a vanishing volume (scale factor) of universe with the infinite energy density and pressure, implying that the Ricci scalar also blows up when the singularity is approached. The null energy condition is satisfied at these singularities. They signal the ultimate future/beginning of a contracting/expanding classical universe which can be reached at a finite proper time along the geodesics.

- 大爆炸/大挤压奇点: 这类奇点的特征是宇宙体积 (标度因子) 趋近于零，能量密度和压强趋于无穷，意味着接近奇点时里奇标量也会发散。这类奇点满足零能量条件，标志着收缩/膨胀经典宇宙的最终未来/开端，沿测地线可在有限固有时内到达。

- Big Rip or Type-I singularity: At this type of singularity, all of the relevant quantities, such as the scalar factor, the energy density, the pressure as well as the Ricci curvature, become divergent. The null energy condition is also violated at the singularity. An example of this singularity can be found in the models with a phantom field of the equation of state $w < -1$. If we consider a phantom fluid with a fixed equation of state, then the conservation law results in $\rho \propto a^{-3(1+w)}$, which shows that the energy density blows up as the scale factor becomes divergent. The pressure and the Ricci scalar have the same fate.

- 大撕裂或 I 型奇点: 这类奇点中，标度因子、能量密度、压强以及里奇曲率等所有相关物理量都会发散。这类奇点也不满足零能量条件，物态方程为 $w < -1$ 的 phantom 场模型中就存在这类奇点。如果考虑物态方程固定的 phantom 流体，守恒律给出 $\rho \propto a^{-3(1+w)}$ ，这表明当标度因子发散时能量密度也会发散，压强和里奇标量也会同样发散。

- Sudden or Type-II singularity: This singularity is featured by a finite scale factor and the energy density. However, both the pressure and the Ricci scalar become infinite when the singularity is reached [107]. Therefore, the dominant energy conditions are violated for this type of singularities.

- 突然奇点或 II 型奇点: 这类奇点的特征是标度因子和能量密度均有限，但到达奇点时压强和里奇标量都会变为无穷 [107]，因此这类奇点不满足主能量条件。

- Big Freeze or Type-III singularity: This type of singularity also occurs at finite value of the scale factor, but all the other physical observables, such as the energy density, the pressure, and the Ricci scalar, become divergent [108].

- 大冻结或 III 型奇点: 这类奇点同样发生在标度因子取有限值时，但能量密度、压强、里奇标量等所有其他物理可观测量都会发散 [108]。

- Type-IV singularity: As compared with all of its cousins discussed above, this type of singularities behaves normally when assessed by the behavior of the scale factor, the energy density, the pressure and the Ricci scalar since all of these quantities are finite valued. The divergence shows up when the curvature derivatives are computed [109].

- IV 型奇点: 和上述所有其他类型的奇点相比，这类奇点在标度因子、能量密度、压强和里奇标量的行为上都表现正常，所有这些量都是有限的，发散出现在曲率导数的计算中 [109]。

The analysis of the energy conditions for the last three types of the singularities is model dependent [104]. Although the singularities discussed above are related with the divergence of physical observables or their derivatives, not all of these singularities can be regarded as the physical ones. Depending on the strength of the tidal forces at the singularities, the singularities can be either the strong singularities or the weak singularities [110-112]. Intuitively speaking, at the strong singularities, the tidal forces are strong enough to destroy any objects or detectors that try to pass through, while the weak singularities still allow the passage of strong detectors. The strength of the singularities can be studied in a quantitative way using the projection of the Ricci tensor along the tangent direction of the geodesics, namely,

后三类奇点的能量条件分析依赖于具体模型 [104]。尽管上述讨论的奇点都和物理可观测量或其导数的发散有关，但并非所有这些奇点都可被视为物理奇点。根据奇点处潮汐力的强度，奇点可分为强奇点和弱奇点 [110-112]。直观来说，强奇点处的潮汐力足够强，能够摧毁任何试图穿过的物体或探测器，而弱奇点允许强度足够大的探测器穿过。可以利用里奇张量沿测地线切线方向的投影对奇点强度进行定量研究，即

$$\int_0^\tau \mathrm{d}\tau R_{ab} u^a u^b. \tag{30}$$

If the above integral becomes divergent when τ asymptotes to the finite proper time at which the singularity occurs, then this singularity is regarded as strong by Królak [112]. If the above integral remains finite, then the singularity is weak. A more restrictive version of the above criterion is to compute the double integral of the same integrand as considered by Tipler [111]. Straightforward calculations of the integral (30) in the FLRW universe reveal that the result of the integral is finite when the scale factor is a finite value if only its second- or higher-order derivatives are divergent. As a result, the big-bang/crunch/rip singularity is the strong singularity, while the Type-II/IV singularities are weak. For the Type-III singularity, it is strong according to Królak's condition and weak by Tipler's condition. It is believed that strong singularities are also the ones where geodesics cannot be extended.

当 τ 渐近趋近于奇点发生的有限固有时，若上述积分发散，则根据 Królak[112] 的定义，该奇点被视为强奇点；若上述积分仍保持有限，则该奇点是弱奇点。Tipler[111] 对该判据提出了一个更严格的版本，即对同一被积函数计算二重积分。对 FLRW 宇宙中的积分 (30) 直接计算表明: 仅当尺度因子的二阶及更高阶导数发散、尺度因子本身为有限值时，积分结果为有限。由此可得，大爆炸/大挤压/大撕裂奇点是强奇点，而 II 型/IV 型奇点是弱奇点。对于 III 型奇点，根据 Królak 判据它是强奇点，根据 Tipler 判据它是弱奇点。普遍认为，强奇点处测地线也无法延拓。

Shapes of the Singularities

奇点的形状

In the isotropic models, the scale factor in each direction behaves in a uniform way, and the singularities that are reached by a vanishing scale factor take the form of a point and thus are called the point singularities.

In contrast, the structure of the singularities becomes much richer in the presence of anisotropies where the directional scale factors can approach the singularities in different manners. Depending on the behavior of the directional scale factor when a singularity is approached, there can be different shapes as listed below [113]:

> 在各向同性模型中，各个方向的标度因子表现均匀，由标度因子趋近于零得到的奇点呈点形，因此被称为点奇点。与之相对，当存在在各向异性时，不同方向的标度因子可以不同方式趋近奇点，奇点的结构会丰富得多。根据趋近奇点时方向标度因子的行为，可分为以下不同的形状 [113]:

- Point singularity: The directional scale factors simultaneously reach zero at the singularities, mimicking the behavior of the point-like singularity in the isotropic model. For this type of singularities to occur, the matter-energy density must dominate over that of the anisotropic shear which implies that the equation of state of the matter must satisfy $w \geq 1$. An example in this case is an anisotropic universe dominated by a single scalar field with a negative potential.

> - 点奇点: 各方向标度因子在奇点处同时归零，模仿了各向同性模型中点状奇点的行为。要形成这类奇点，物质能量密度必须占各向异性剪切的主导，这要求物质的物态方程满足 $w \geq 1$ 。这类情况的一个例子是由负势单标量场主导的各向异性宇宙。

- Barrel singularity: This singularity appears when one of the directional scale factors tends to a nonzero constant value, while the other two become vanishing.

> - 桶形奇点: 这类奇点出现于一个方向标度因子趋于非零常数，另外两个方向标度因子趋近于零的情况。

- Pancake singularity: In contrast to the Barrel singularity, the Pancake singularity is characterized by the vanishing of one of the directional scale factors with the other two approaching nonzero constants.

> - 煎饼形奇点: 与桶形奇点相反，煎饼形奇点的特征是一个方向标度因子趋近于零，另外两个方向标度因子趋于非零常数。

- Cigar singularity: This singularity emerges when one of the directional scale factors becomes divergent while the other two tend to vanish.

> - 雪茄形奇点: 这类奇点出现于一个方向标度因子发散，另外两个方向标度因子趋近于零的情况。

In the presence of anisotropies, a point-like singularity can occur only when matter-energy density dominates evolution over the anisotropic shear. Since the anisotropic shear behaves as $\sigma^2 \propto a^{-6}$, this means that for all matter with equation of state $w < 1$, the shape of the singularity is not point-like. In general, the singularity is a cigar-like singularity. In particular, when the matter is absent, only the pancake and the cigar singularities can occur in the Bianchi-I spacetime. This can be easily understood by introducing the Kasner exponents k_c with $c = 1, 2, 3$ in which the directional scale factors of the classical solutions scale as $a_c \propto t^{k_c}$. For the vacuum solution, there are two constraints on the Kasner exponents, namely $k_1 + k_2 + k_3 = 1$ and $k_1^2 + k_2^2 + k_3^2 = 1$. As a result, point singularity is impossible to occur since it requires $k_1 = k_2 = k_3 = 1/3$, which violates the second condition. For the same reason, neither the barrel singularity which requires

$k_1 = 0, k_2 = k_3 = 1/2$ is possible. In contrast, the pancake singularity can be reached by $k_1 = k_2 = 0, k_3 = 1$, while the cigar singularity can be realized by choosing one of the Kasner exponents to be a small negative number and then solve for the other two. Thus, the allowed shapes of the singularity are essentially determined by the initial conditions on the matter content and anisotropies. When the Bianchi-I universe is filled with dust, only the cigar and pancake singularities are possible to occur. In contrast, if the matter content is composed of stiff matter, then the barrel, cigar, and point singularities can form.

存在各向异性时，只有当物质能量密度在演化中主导各向异性剪切，才会形成点状奇点。由于各向异性剪切的行为满足 $\sigma^2 \propto a^{-6}$ ，这意味着对于所有物态方程满足 $w < 1$ 的物质，奇点形状都不是点状的，一般会形成雪茄形奇点。特别地，当不存在物质时，Bianchi-I 时空仅能形成煎饼形奇点和雪茄形奇点。引入卡斯纳指数 k_c (满足 $c = 1, 2, 3$) 后很容易理解这一点: 经典解的方向标度因子按 $a_c \propto t^{k_c}$ 标度。对于真空解，卡斯纳指数存在两个约束，即 $k_1 + k_2 + k_3 = 1$ 和 $k_1^2 + k_2^2 + k_3^2 = 1$ 。因此，点奇点不可能出现，因为它要求 $k_1 = k_2 = k_3 = 1/3$ ，违反了第二个约束条件。同理，要求 $k_1 = 0, k_2 = k_3 = 1/2$ 的桶形奇点也不可能出现。与之相对，$k_1 = k_2 = 0, k_3 = 1$ 可以形成煎饼形奇点，而将一个卡斯纳指数取为小负数再求解另外两个，就能得到雪茄形奇点。因此，奇点的允许形状本质上由物质组分和各向异性的初始条件决定。当 Bianchi-I 宇宙充满尘埃时，仅能形成雪茄形和煎饼形奇点。与之相对，如果物质组分为刚性物质，则可以形成桶形、雪茄形和点奇点。

Having discussed the nature of classical singularities in some detail, it is pertinent to ask whether quantum geometry effects as understood in LQC resolve all different types of singularities, and what happens to the shape of the bounce. To answer these questions, we need to go beyond the classical setting and perform a loop quantization of the isotropic and anisotropic models.

在详细讨论了经典奇点的性质之后，有一个相关问题: 圈量子引力 (LQC) 框架下理解的量子几何效应是否能解决所有不同类型的奇点？反弹之后奇点的形状会发生什么变化？要回答这些问题，我们需要超越经典框架，对各向同性和各向异性模型进行圈量子化。

Loop Quantum Cosmology: Spatially Flat Isotropic Model

圈量子宇宙学: 空间平坦各向同性模型

In this section, we overview the key steps of the loop quantization of a spatially flat FLRW universe as originally proposed in [8-10]. It provides a framework where all the aspects concerning the construction of a quantum theory of gravity, such as the quantum Hamiltonian constraint, the physical Hilbert space, the inner product, the Dirac observables, and the semiclassical states, are obtained systematically. The construction of this model also serves as a prototype for applying the techniques of LQG to symmetry reduced spacetimes with the purpose of understanding the singularity resolution and the effects of quantum gravity therein. Later, it was succeeded by a series of remarkable works exploring the quantum gravity effects in the spacetimes with more complicated structure, such as those with a cosmological constant [114-116], radiation [78], the spatial curvature [55, 56, 117, 118], the anisotropies [58-61, 119-121], in the presence of inhomogeneities [62, 64-67], as well as other variants resulting from different quantization prescriptions [74]. Although the loop quantization of these different cosmological models exhibits their own properties in the Planck regime, there also exists one common feature originating from introducing the minimal area gap in LQG in the course of quantization, that is, the generic resolution of the spacetime curvature singularities [23-27,76,122]. In the following, we start with the main steps of the loop quantization of the spatially flat FLRW universe. A key step in this quantization procedure deals with a careful assignment of the area of the loop over which holonomies are considered. In the early stage of LQC, this area was taken to be the coordinate area, which resulted in a quantum theory that suffers from several drawbacks, for example the dependence of the bounce density on the initial conditions and the fiducial cell and lack of infrared limit as GR when matter violates strong energy condition [123]. It turns out that the quantization emerging from considering physical areas, also known as improved dynamics or $\bar{\mu}$ scheme in LQC [8], is the only physically viable choice in the isotropic models [123, 124]. In the improved dynamics of LQC, it is more convenient to work with b and v variables obtained from the c and p variables using a canonical transformation (see (16)) [15]. In the following, we first discuss the $\bar{\mu}$ scheme in LQC using volume representation and then comment on the uniqueness of the $\bar{\mu}$ scheme.

本节我们概述最初由文献 [8-10] 提出的空间平坦 FLRW 宇宙的圈量子化关键步骤。该模型提供了一个框架，可系统得到量子引力构造的所有相关内容，包括量子哈密顿约束、物理希尔伯特空间、内积、狄拉克可观测量和半经典态。该模型的构造也可作为原型，用于将 LQG 技术应用到对称约化时空，以研究奇异性消解和其中的量子引力效应。后续，一系列重要工作在此基础上拓展，探究了结构更复杂的时空中的量子引力效应，例如带宇宙常数的时空 [114-116]、辐射时空 [78]、带空间曲率的时空 [55, 56, 117, 118]、各向异性时空 [58-61, 119-121]、存在非均匀性的时空 [62, 64-67]，以及由不同量子化方案得到的其他变体模型 [74]。尽管这些不同宇宙学模型的圈量子化在普朗克能区呈现出各自的性质，但量子化过程中引入 LQG 的最小面积间隙会带来一个共同特征，即时空曲率奇异性的普遍消解 [23-27,76,122]。下文我们将从空间平坦 FLRW 宇宙圈量子化的主要步骤开始介绍。该量子化过程的一个关键步骤是仔细确定全同调所在圈的面积。在 LQC 发展早期，该面积被取为坐标面积，由此得到的量子理论存在多个缺陷，例如反弹密度依赖于初始条件和参考格胞，且当物质违反强能量条件时不存在广义相对论的红外极限 [123]。结果表明，考虑物理面积得到的量子化，也就是 LQC 中的改进动力学即 $\bar{\mu}$ 方案 [8]，是各向同性模型中唯一符合物理要求的选择 [123, 124]。在 LQC 的改进动力学中，使用通过正则变换从 c 和 p 变量得到的 b 和 v 变量会更方便 (见式 (16))[15]。下文我们首先利用体积表示讨论 LQC 中的 $\bar{\mu}$ 方案，再对 $\bar{\mu}$ 方案的唯一性进行评述。

Loop Quantization of the Spatially Flat FLRW Universe

空间平坦 FLRW 宇宙的圈量子化

In this subsection, applying the techniques in full LQG, we outline the construction of the Hamiltonian constraint operator for the spatially flat FLRW universe in LQC as originally developed in [9,10]. The fundamental variables for loop quantization in LQG are the holonomies of the connection along a path l and the fluxes of the triads over a 2-surface S , and to be specific, they are formally given by

在本小节中，我们应用完整圈量子引力 (LQG) 的方法，概述如文献 [9,10] 最初构建的圈量子宇宙学 (LQC) 中空间平坦 FLRW 宇宙的哈密顿约束算子构造过程。LQG 中圈量子化的基本变量是联络沿路径 l 的全纯联络，以及三维标架在二维曲面上的流 S ，具体形式可写为

$$h_l = \mathcal{P}\exp\left(\int_l \mathrm{d}l A^i_a \tau_i \frac{\mathrm{d}x^a}{\mathrm{d}l}\right),\ E_i = \int_S n_a E^a_i\, \mathrm{d}^2 s, \tag{31}$$

where the path is parameterized by $x^a(l)$, $\mathcal{P}$ stands for the path-ordered product, n_a is the unit normal to the surface S , and $\tau_i = -i\sigma_i/2$ with σ_i being the Pauli matrices. In the spatially flat FLRW universe, due to the homogeneity and isotropy of the spacetime, the integrals in the above definitions can be computed analytically. For the holonomies, we consider a path along the edge $\mu \overset{\circ}{e}{}_i^a$, yielding

其中路径由参数 $x^a(l)$, $\mathcal{P}$ 描述，\$P\$ 代表路径排序乘积，n_a 是曲面 S 的单位法向量，且 $\tau_i = -i\sigma_i/2$ 中 σ_i 为泡利矩阵。在空间平坦 FLRW 宇宙中，由于时空具有均匀各向同性，上述定义中的积分可以解析求解。对于全纯联络，我们考虑沿边 $\mu \overset{\circ}{e}{}_i^a$ 的路径，可得

$$h_i^{(\mu)} = \exp \int_0^{V_0^{1/3}} \tau_j A_a^j \mu \mathring{e}_i^a \, \mathrm{d}l = \cos\left(\frac{\mu c}{2}\right)\mathbb{I} + 2\tau_i \sin\left(\frac{\mu c}{2}\right), \tag{32}$$

where the symmetry reduced connection (15) is used. Similarly, one can explicitly work out the fluxes of the triads which turn out to be proportional to the symmetry reduced triad p [12]. Therefore, the elementary classical variables of the geometric sector for loop quantization of a spatially flat FLRW universe are the almost periodic functions of the connection $N_\mu = e^{i\mu c/2}$ and the isotropic triad p . They form the so-called abstract $\star$ -algebra which has a unique representation in the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$: $L^2(\mathbb{R}_{\text{Bohr}}, \mathrm{d}\mu_{\text{Bohr}})$ [125-127]. $\mathcal{H}_{\text{kin}}$ consists of the square integrable functions on the Bohr compactification of the real line with the basis states given by $N_\mu := \langle c \mid \mu \rangle$. The label of the states μ is essentially discrete as can be seen from the inner product of two basis states

此处使用了对称性约化后的联络 (15)。同理，可以直接计算得到三维标架的流，结果与对称性约化后的三维标架 p 成正比 [12]。因此，空间平坦 FLRW 宇宙圈量子化几何部分的基本经典变量是联络 $N_\mu = e^{i\mu c/2}$ 的概周期函数和各向同性三维标架 p 。它们构成了所谓的抽象 $\star$ 代数，该代数在运动学希尔伯特空间 $\mathcal{H}_{\text{kin}}$: $L^2(\mathbb{R}_{\text{Bohr}}, \mathrm{d}\mu_{\text{Bohr}})$ 中存在唯一表示 [125-127]。$\mathcal{H}_{\text{kin}}$ 由实数线玻紧化上的平方可积函数构成，基态由 $N_\mu := \langle c \mid \mu \rangle$ 给出。从两个基态的内积可以看出，态的标号 μ 本质上是离散的

$$\langle \mu_1 \mid \mu_2 \rangle = \lim_{D\to\infty} \frac{1}{2D} \int_{-D}^{D} \mathrm{d}c \, \langle \mu_1 \mid c \rangle \langle c \mid \mu_2 \rangle = \delta_{\mu_1,\mu_2}, \tag{33}$$

where δ_{μ_1,μ_2} is the Kronecker delta. A general state in $\mathcal{H}_{\text{kin}}$ is a countable sum of the basis states as $\left|\Psi\right\rangle = \sum_n \alpha_n \left|\mu_n\right\rangle$ with the complex coefficients satisfying the condition $\sum_n |\alpha_n|^2 < \infty$. There are unambiguous operator representations of N_μ and the triad p which act on the basis states in the way [9]

其中 δ_{μ_1,μ_2} 是克罗内克 delta 函数。$\mathcal{H}_{\text{kin}}$ 中的任意一般态都可表示为基态的可数求和 $\left|\Psi\right\rangle = \sum_n \alpha_n \left|\mu_n\right\rangle$ ，其复系数满足条件 $\sum_n |\alpha_n|^2 < \infty$ 。N_μ 和三维标架 p 都有明确的算子表示，它们对基态的作用为 [9]

$$\widehat{N}_{\mu_0} |\mu\rangle = |\mu + \mu_0\rangle, \ \widehat{p}|\mu\rangle = \frac{8\pi\gamma l_{\text{pl}}^2}{6}\mu|\mu\rangle, \tag{34}$$

here μ_0 is a constant, and the Planck length $l_{\text{pl}}^2 = G\hbar$. From the operator $\widehat{N}_\mu$, one can easily find the action of the holonomy operator $\widehat{h}_i^{(\mu)}$ on the states in $\mathcal{H}_{\text{kin}}$ [10]. The operators $\widehat{h}_i^{(\mu)}$ and $\widehat{p}$ are the elementary building blocks for constructing the quantum Hamiltonian constraint in LQC. Now it is time to take the advantage of the symmetry of the spacetime under consideration, and due to homogeneity and isotropy, it is obvious that the Gauss and the momentum constraints given in (7) vanish identically. Therefore, only the

Hamiltonian constraint (8) remains to be implemented. Furthermore, the Lorentzian term in the Hamiltonian constraint (8) turns out to be a multiple of the Euclidean term in the spatially flat FLRW universe. As a result, in standard LQC, the geometrical sector of the quantum Hamiltonian constraint is constructed from its classical counterpart

其中 μ_0 是常数，${l}_{ \text{Pl}}$ 是普朗克长度 $l_{\rm pl}^2 = G\hbar$ 。从算子 $\widehat{N}_\mu$ 可以很容易得到全纯联络算子 $\widehat{h}_i^{(\mu)}$ 对 $\mathcal{H}_{\rm kin}$ 中态的作用 [10]。算子 $\widehat{h}_i^{(\mu)}$ 和 $\widehat{p}$ 是构造 LQC 量子哈密顿约束的基本构建块。现在我们可以利用所研究时空的对称性，由于均匀各向同性，(7) 给出的高斯约束和动量约束显然恒为零，因此仅剩余哈密顿约束 (8) 需要量子化。此外，在空间平坦 FLRW 宇宙中，哈密顿约束 (8) 中的洛伦兹项恰好是欧几里得项的倍数。因此，在标准 LQC 中，量子哈密顿约束的几何部分是由其经典对应构造而来

$$\mathcal{H}_{\rm g} = -\int_{\mathcal{V}} \mathrm{d}^3x \frac{N}{2\kappa\gamma^2\sqrt{|q|}} E_j^a E_k^b \varepsilon_i^{\ jk} F_{ab}^i, \tag{35}$$

which absorbs the Lorentzian term into the coefficient of the Euclidean term. A separate treatment of the Lorentzian term can result in phenomenologically distinct models [74, 77] as will be discussed in section "Beyond Standard LQC: Incorporating Additional Elements from LQG". Next we need to express the above Hamiltonian in terms of the holonomies and the triad. It contains two pieces, the first piece can be written as [6, 128]

它将洛伦兹项吸收到欧几里得项的系数中。单独处理洛伦兹项会得到唯象上不同的模型 [74, 77]，我们会在“标准圈量子宇宙学之外: 纳入 LQG 的额外元素”一节对此展开讨论。接下来我们需要将上述哈密顿量用全纯和三重表述出来。它分为两部分，第一部分可以写为 [6, 128]

$$\frac{\varepsilon_{ijk}}{\sqrt{|q|}} E^{aj} E^{bk} = \sum_k \frac{\mathrm{sgn}\,(p)}{2\pi\gamma G\mu V_o^{1/3}} \mathring{\varepsilon}^{abc} \mathring{\omega}_c^k \,\mathrm{Tr}\left(h_k^{(\mu)} \left\{ \left(h_k^{(\mu)}\right)^{-1}, V \right\} \tau_i \right), \tag{36}$$

where $V = |p|^{3/2}$ is the physical volume and "Tr" means to take the trace of the parenthesis. The other piece is the field strength F_{ab}^i which can be expressed in terms of the holonomies over a square in the $j-k$ plane spanned by the triads $\mathring{e}_j^a$ and $\mathring{e}_k^b$, namely,

其中 $V = |p|^{3/2}$ 是物理体积，“Tr”表示对括号内的量取迹。另一部分是场强 F_{ab}^i ，它可以用由三重 $\mathring{e}_j^a$ 和 $\mathring{e}_k^b$ 张成的 $j-k$ 平面上正方形的全纯表示，即:

$$F_{ab}^i = -2 \lim_{\mu\to 0} \frac{\mathrm{Tr}\left[\tau^i \left(h_{\square_{jk}}^{(\mu)} - \mathbb{I}\right)\right]}{\mu^2 V_0^{2/3}} \mathring{\omega}_a^j \mathring{\omega}_b^k, \tag{37}$$

where $h_{\square_{jk}}^{(\mu)} = h_j^{(\mu)} h_k^{(\mu)} \left(h_j^{(\mu)}\right)^{-1} \left(h_k^{(\mu)}\right)^{-1}$ is the holonomy over a square loop. In the classical theory, the field strength is a local quantity as the area of the square vanishes when $\mu \to 0$. When shifting to the quantum theory, the holonomies and the volume in (36)-(37) should be promoted to their operator analogs and the Poisson brackets be replaced by the commutators. In addition to these standard operations, extra care should be taken of the area of the square used to define the field strength.

其中 $h^{(\mu)}_{\square_{jk}} = h^{(\mu)}_j h^{(\mu)}_k \left(h^{(\mu)}_j\right)^{-1} \left(h^{(\mu)}_k\right)^{-1}$ 是方形闭合回路的全纯。经典理论中，当 $\mu \to 0$ 时正方形面积趋于零，因此场强是一个局域量。过渡到量子理论时，式 (36)-(37) 中的全纯和体积都需要提升为对应的算符，泊松括号替换为对易子。除这些标准操作外，还需要额外注意用来定义场强的正方形的面积。

The key point is that in LQG, the area operator has a discrete spectrum with a minimal nonzero eigenvalue. This feature is inherited in LQC in the sense that a nonlocal field strength operator is defined by shrinking the area of the square to the minimal area gap available for the homogeneous spacetime which is usually denoted by $\Delta = 4\sqrt{3}\pi\gamma l_{\mathrm{pl}}^2$. Relating the physical area of the square $\left(a^2\mu^2 V_o^{2/3}\right)$ with this minimal area gap, we can fix the value of μ to be

关键在于，在圈量子引力 LQG 中，面积算符具有离散谱，且存在最小非零本征值。这一性质被 LQC 继承: 对于齐次时空，我们将正方形的面积收缩到可用的最小面积间隙 (通常记为 $\Delta = 4\sqrt{3}\pi\gamma l_{\mathrm{pl}}^2$),以此定义非局域场强算符。将正方形的物理面积 $\left(a^2\mu^2 V_o^{2/3}\right)$ 与这个最小面积间隙关联后，我们可以将 μ 的值固定为

$$\mu = \bar{\mu} \ := \ \frac{\lambda}{\sqrt{|p|}} \tag{38}$$

with $\lambda = \sqrt{\Delta}$. This choice of μ leads to the well-known $\bar{\mu}$ scheme in LQC. Since $\bar{\mu}$ explicitly depends on the triad, $\widehat{N}_{\bar{\mu}}$ no longer acts as a shift operator on the eigenstates of the triad operator $\hat{p}$. In order to make the quantum Hamiltonian constraint operator a difference operator with uniform step size, it is convenient to work with the eigenstates of the volume operator on which $\widehat{N}_{\bar{\mu}}$ behaves like a shift operator again. Specifically, we have [10, 129]

其中 $\lambda = \sqrt{\Delta}$ 。这一 μ 的选择引出了 LQC 中著名的 $\bar{\mu}$ 方案。由于 $\bar{\mu}$ 显式依赖于三重，$\widehat{N}_{\bar{\mu}}$ 不再是三重算符 $\hat{p}$ 本征态上的平移算符。为了让量子哈密顿约束算符成为步长均匀的差分算符，使用体积算符的本征基会更方便，此时 $\widehat{N}_{\bar{\mu}}$ 恢复为平移算符。具体来说，我们有 [10, 129]

$$\hat{V}\Big|v\rangle = \left(\frac{8\pi\gamma}{6}\right)^{3/2} \frac{|v|}{K} l_{\mathrm{pl}}^3 \Big| v\rangle,\ \widehat{N}_{\bar{\mu}}\,|v\rangle = | \, v+1\rangle, \tag{39}$$

where $K = \frac{2}{3\sqrt{3\sqrt{3}}}$ and two sets of basis states, i.e., $|v\rangle$ and $|\,\mu\rangle$ are related with each other via $v = K\,\mathrm{sgn}\,(\mu)\,|\mu|^{3/2}$. To extract dynamics, we consider a massless scalar field which serves as a matter clock, whose classical and Fock quantized Hamiltonian is given by

其中 $K = \frac{2}{3\sqrt{3\sqrt{3}}}$ 和两组基矢，即 $|v\rangle$ and $|\,\mu\rangle$ ，可以通过 $v = K\,\mathrm{sgn}\,(\mu)\,|\mu|^{3/2}$ 相互变换。为了提取动力学，我们考虑无质量标量场作为物质时钟，它的经典哈密顿量和福克量子化哈密顿量如下:

$$\mathcal{H}_{\mathrm{m}} = \frac{p_\phi^2}{2|p|^{3/2}} \to \widehat{\mathcal{H}}_{\mathrm{m}} = -\frac{\widehat{|p|^{-3/2}}\partial^2}{2\partial\phi^2}. \tag{40}$$

Note that unlike the gravitational sector, the Schrödinger representation is employed for the scalar field. Now according to the Dirac's quantization approach for the constrained system, the physical state $\Psi\,(v,\phi)\ :=$

$\langle \Psi \mid v \rangle$ must be annihilated by the quantum Hamiltonian constraint operator, that is,

注意和引力部分不同，标量场采用薛定谔表示。根据狄拉克对约束系统的量子化方法，物理态 $\Psi(v,\phi) := \langle \Psi \mid v \rangle$ 必须被量子哈密顿约束算符湮灭，即:

$$\widehat{\mathcal{H}}_0 \Psi(v,\phi) = \left(\widehat{\mathcal{H}}_{\mathrm{g}} + \widehat{\mathcal{H}}_{\mathrm{m}}\right) \Psi(v,\phi) = 0. \tag{41}$$

Combining (35), (36), (37), (40), and (41), we can obtain a quantum difference equation which governs the dynamical evolution of the physical state and reads explicitly [10]

结合 (35)、(36)、(37)、(40) 和 (41)，我们可以得到支配物理态动力学演化的量子差分方程，其具体形式为 [10]

$$\partial_\phi^2 \Psi(v,\phi) = [B(v)]^{-1} \left(C^+ \Psi(v+4,\phi) + C^0 \Psi(v,\phi) + C^- \Psi(v-4,\phi)\right)$$

$$=: -\Theta \Psi(v,\phi), \tag{42}$$

where the volume-dependent coefficients are given by

其中依赖体积的系数由下式给出

$$B(v) = \left(\frac{3}{2}\right)^3 K |v| \left| |v+1|^{1/3} - |v-1|^{1/3} \right|^3, C^+(v)$$

$$= \frac{3\pi KG}{8} |v+2| \,\|\, v+1 \,||\, v+1 \,|-|\, v+3 \,\|,$$

$$C^-(v) = C^+(v-4), \ C^0(v) = -C^+(v) - C^-(v). \tag{43}$$

The quantum Hamiltonian constraint equation (42) prescribes the quantum evolution of the physical states with respect to the emergent time ϕ. Unlike its counterpart in the Wheeler-DeWitt theory which is a differential equation on a continuum background spacetime [10, 15], Eq. (42) is a second-order difference equation with a uniform spacing in the volume. It naturally divides the space of the physical states into a set of superselection sectors with support on the lattices: $\mathcal{L}_{\pm\varepsilon} = \{v = \pm(4n+\varepsilon)\}$ with $n \in \mathbb{N}$ and $\varepsilon \in (0,4]$, and the physical states defined in different sectors will not be mixed over time by the evolution equation. Moreover, using the group averaging techniques [130-133], we can define the inner product of the physical states and the relevant Dirac observables, such as the volume operator at a particular time $\hat{v}_{\phi_0}$ and the momentum operator $\widehat{p}_\phi$ [10]. These Dirac observables also preserve the superselection sectors, and their expectation values under the physical states can be used to extract the physical predictions of the quantum theory. Furthermore, physical semiclassical states can be constructed precisely in LQC, and the numerical simulations of their dynamical evolution under quantum difference equation (42) lead to quantitative results on the singularity resolution. The evolution of different types of the physical states, starting from the sharply peaked Gaussian states to the highly squeezed and non-Gaussian states, has been extensively studied in order to test the robustness of the singularity resolution [10,13,134,135]. For the sharply peaked Gaussian states, it has been found that these states remain peaked in the backward evolution during which the volume of the universe shrinks. At very small spacetime curvature, the evolution of the expectation value of the volume operator coincides with the classical trajectories to a great accuracy. Deviations of the quantum evolution from the classical trajectories occur at high spacetime curvature where the expectation value of the volume operator in the quantum theory bounces at the critical energy density $\rho_c \approx 0.41\rho_{\rm Pl}$. After the bounce, the volume of the universe starts to increase again. The quantum geometry effects become dominant only in a small neighborhood of the bounce which connects the contracting branch with the expanding one. This feature of the quantum dynamics in LQC can also be precisely captured by the effective dynamics of LQC based on a continuum spacetime description as will be discussed in detail in section ”The Effective Dynamics of Isotropic Model”. In Fig. 1, with the massless scalar field as an emergent time, we compare the trajectory of the expectation value of the volume operator with those from the effective dynamics of LQC as well as the classical GR. It is obvious from the figure that the effective dynamics gives an accurate approximation of the expectation value of the volume operator from the quantum dynamics throughout the whole evolution of the universe, including in particular the Planckian regime. In addition to the highly peaked states, the quantum bounce is also found to be a robust feature for the widely spread or squeezed states for which the bounce will occur at a lower maximum energy density than ρ_c [13, 134] , as well as different quantization prescriptions in LQC [136, 137] and even when more features from full LQG are incorporated into the construction of the model (see section ”Beyond Standard LQC: Incorporating Additional Elements from LQG” for more details).

量子哈密顿约束方程 (42) 规定了物理态相对于涌现时间 ϕ 的量子演化。惠勒-德维特理论中的对应方程是连续背景时空上的微分方程 [10, 15]，与之不同，方程 (42) 是体积上间距均匀的二阶差分方程。它自然将物理态空间划分为支集在格点上的一系列超选择分支: $\mathcal{L}_{\pm\varepsilon} = \{v = \pm(4n + \varepsilon)\}$ (满足 $n \in \mathbb{N}$ 和 $\varepsilon \in (0, 4]$)，不同分支定义的物理态不会在演化过程中随时间混合。此外，利用群平均技术 [130-133]，我们可以定义物理态的内积以及相关狄拉克可观测量，例如特定时间 $\hat{v}_{\phi_0}$ 处的体积算符和动量算符 $\hat{p}_\phi$ [10]。这些狄拉克可观测量也保持超选择分支不变，它们在物理态下的期望值可用于提取量子理论的物理预言。进一步来说，LQC 中可以精确构造物理半经典态，对它们在量子差分方程 (42) 下动力学演化的数值模拟得到了奇点解决的定量结果。为了检验奇点解决的稳健性，人们已经广泛研究了从尖锐峰高斯态到高度压缩非高斯态等不同类型物理态的演化 [10,13,134,135]。对于尖锐峰高斯态，研究发现这类态在体积收缩的逆演化过程中始终保持峰形。在极低时空曲率下，体积算符期望值的演化与经典轨迹高度吻合。当时空曲率很高时，量子演化偏离经典轨迹，量子理论中体积算符的期望值会在临界能量密度 $\rho_c \approx 0.41\rho_{\rm Pl}$ 处发生反弹。反弹后，宇宙体积再次开始膨胀。量子几何效应仅在连接收缩分支和膨胀分支的反弹邻域内占主导。正如“各向同性模型的有效动力学”一节会详细讨论的，基于连续时空描述的 LQC 有效动力学也能精确捕捉 LQC 量子动力学的这一特征。图 1 中，我们以无质量标量场作为涌现时间，对比了体积算符期望值的轨迹与 LQC 有效动力学、经典广义相对论的轨迹。从图中可明显看出，有效动力学在宇宙整个演化过程中，尤其是普朗克尺度区域，都能准确近似量子动力学给出的体积算符期望值。除了高峰度态外，对于广延分布或压缩态，量子反弹也是稳健的特征——这类态的反弹发生在低于 ρ_c [13, 134] 的最大能量密度处；而且 LQC 中不同量子化方案下 [136, 137]，甚至在模型构造中纳入完整 LQG 更多特征后 (详见“超越标准 LQC: 纳入 LQG 额外要素”一节)，量子反弹依然存在。

The robustness of the singularity resolution has also been further confirmed using analytical solutions in the solvable loop quantum cosmology (sLQC) [15]. This model is obtained by recasting LQC in the spatially flat FLRW universe with a massless scalar field in the b representation. Due to the availability of the analytical expressions of the physical states, one can explicitly compute the expectation values of the volume and the energy density operators for an arbitrary physical state. It turns out that the expectation values of the volume operator have a nonzero minimum which implies the existence of the bounce. Furthermore, the computation of the expectation values of the energy density reveals an upper bound exactly equal to $0.41\rho_{\rm Pl}$ which is exactly the same as the maximum energy density encountered in the numerical simulations and the effective dynamics. In addition to confirming the robustness of the bounce for an arbitrary state in the physical Hilbert space of LQC, sLQC also plays important roles in various aspects in the investigations of the related subjects, such as understanding the evolution of quantum fluctuations across the bounce [16, 17, 138, 139], computing the quantum probabilities for the occurrence of the bounce in LQC [18] and singularities in Wheeler-DeWitt theory [20] as well as exploring the relations between LQC and the Wheeler-DeWitt theory [15]. Here let us note that the consistent quantum probability for singularity to occur in the above model in LQC turns out to be zero, while the bounce has the probability of unity [18]. On the other hand, the probability for singularity to occur in Wheeler-DeWitt quantization of the same model is unity [20].

借助可解圈量子宇宙学 (sLQC) 中的解析解，奇点消解的鲁棒性也得到了进一步证实 [15]。该模型是将存在无质量标量场的空间平直 FLRW 宇宙中的 LQC 重新表述在 b 表象中得到的。由于可以得到物理态的解析表达式，我们能够对任意物理态显式计算体积算符和能量密度算符的期望值。结果表明，体积算符的期望值存在非零最小值，这意味着反弹必然存在。此外，对能量密度期望值的计算显示能量密度存在恰好等于 $0.41\rho_{\rm Pl}$ 的上限，这一结果与数值模拟和有效动力学中得到的最大能量密度完全一致。除了证实 LQC 物理希尔伯特空间中任意态的反弹都具有鲁棒性外，sLQC 在相关课题研究的诸多方面也发挥着重要作用，例如理解量子涨落在穿越反弹过程中的演化 [16, 17, 138, 139]、计算 LQC 中反弹发生 [18] 与惠勒-德维特理论中奇点发生 [20] 的量子概率，以及探究 LQC 与惠勒-德维特理论之间的关系 [15]。在此需要指出，上述 LQC 模型中奇点发生的一致量子概率为零，而反弹发生的概率为一 [18]；与之相对，同一模型在惠勒-德维特量子化中奇点发生的概率为一 [20]。

Before moving onto the next subsection, a few remarks are in order. Firstly, the mathematical structure of LQC is similar to the polymer quantization of a nonrelativistic quantum mechanical system which is implemented by realizing the Weyl algebra in the kinematic Hilbert space. The latter is unitarily inequivalent to the Schrödinger representation in the sense that one of the canonical pairs is essentially discrete in the polymer representation and there is no operator corresponding to its conjugate variable. In this way, the von-Neumann theorem is bypassed, resulting in the essential difference between the polymer quantization and the Schrödinger quantization [140]. Secondly, LQC adopts a hybrid quantization approach, that is, the gravitational part of the classical Hamiltonian constraint is loop quantized, while the matter sector is quantized in the Schrödinger representation. Therefore, these two sectors are essentially not treated on the same footing. Since deep in the Planck regime, the matter field also lives on the quantum discretized spacetime, the study of the quantum geometry effects on the matter sector is required for a consistent description of the quantum dynamics. Previous works on the polymer quantization of the scalar field and its cosmological implications can be found in a series of papers [141-143]. Further, to obtain the final expression of the quantum difference equation, a unique factor ordering has been chosen for the quantum Hamiltonian constraint operator. There are different choices of the factor ordering as discussed in [9]. The resolution of the curvature singularity is found to be robust against such a choice. Finally, while in this section we have considered the $\overline{\mu}$ scheme for loop quantization arising from the choice (38), in the early literature of LQC, a different choice was made where μ is a constant. It turns out that such a choice runs into various problems, such as with the infrared limit of the theory [123] and occurrence of bounce at a scale which depends on the fiducial cell [10]. In fact, if one considers any other variation of (38) where $\overline{\mu}$ is a function of triad only, such as in lattice refined models considered in [144], one finds that the quantization scheme is severely limited phenomenologically when one considers different matter satisfying null energy condition [123]. In this sense, the $\overline{\mu}$ scheme turns out to be a unique favored choice. This conclusion is also supported by studies on the stability of the quantum difference equation [145, 146] and the unique factor ordering in the continuum limit of LQC [147].

进入下一小节之前，我们先做几点说明。首先，LQC 的数学结构与非相对论量子力学系统的聚合物量子化类似，后者通过在运动学希尔伯特空间中实现外尔代数完成量子化。聚合物量子化与薛定谔表象幺正不等价: 在聚合物表象中，一对正则变量中的一个本质上是离散的，且不存在对应其共轭变量的算符。通过这种方式绕开了冯·诺依曼定理，也使得聚合物量子化与薛定谔量子化存在本质区别 [140]。其次，LQC 采用混合量子化方法: 经典哈密顿约束的引力部分采用圈量子化，而物质部分则在薛定谔表象中量子化，因此这两个部分本质上并没有被同等处理。由于在深普朗克能标区域，物质场也存在于量子离散化的时空上，要对量子动力学给出自洽描述，就需要研究物质部分的量子几何效应。关于标量场聚合物量子化及其宇宙学意义的前期研究可见系列论文 [141-143]。此外，为了得到量子差分方程的最终表达式，我们为量子哈密顿约束算符选择了一种唯一的因子排序。正如文献 [9] 所讨论，因子排序存在多种不同选择，但研究发现曲率奇点的消解对这类选择保持鲁棒性。最后，虽然本节我们讨论了由选择 (38) 得到的 $\bar{\mu}$ 圈量子化方案，但在 LQC 的早期文献中，研究者曾做出过不同选择，其中 μ 为常数。结果表明这种选择会产生诸多问题，例如理论的红外极限问题 [123]，以及反弹发生在依赖于 fiducial 胞元的能标上 [10]。事实上，如果考虑 (38) 的其他变体，其中 $\bar{\mu}$ 仅为三分量量的函数，例如文献 [144] 讨论的格点细化模型，那么当我们考虑满足零能量条件的不同物质时，会发现这种量子化方案唯象上存在极大局限 [123]。从这个意义上说，$\bar{\mu}$ 方案是唯一受青睐的选择，这一结论也得到了量子差分方程稳定性研究 [145, 146] 以及 LQC 连续极限下唯一因子排序研究 [147] 的支持。

The Effective Dynamics of Isotropic Model

各向同性模型的有效动力学

In the previous section we saw that the quantum evolution in LQC is governed by a non-singular quantum difference equation which couples the wavefunction in uniform steps of four Planck volumes in the isotropic model. In general, extracting physical predictions using this equation requires supercomputing resources. However, it is possible to obtain an effective spacetime description (Unlike the usage of this term ”effective” in standard quantum field theory where one integrates out high-energy modes, in the effective spacetime description of LQC, one retains quantum geometric Planck-scale effects. The resulting differential equations from the effective Hamiltonian in the $\bar{\mu}$ scheme of LQC are not classical unless one probes small spacetime curvature regime.) capturing this dynamics for a suitable choice of coherent states using geometric formulation of quantum mechanics. At first sight it may seem naturally puzzling on how can one capture the non-perturbative quantum gravity effects resulting from quantum geometry in a continuum effective spacetime description. The central result from which one can derive this description is that Schrödinger equation is nothing but the Hamilton’s equation in the quantum phase space [148,149]. While a detailed discussion of this approach is beyond the scope of this chapter, we provide a brief glimpse of the underlying idea and refer the reader to [148-153] for more details.

在上一节中我们已经看到，圈量子宇宙学 (LQC) 的量子演化由一个非奇异量子差分方程控制，在各向同性模型中，该方程以四个普朗克体积的均匀步长耦合波函数。一般来说，使用这个方程提取物理预言需要超级计算资源。不过，借助量子力学的几何表述，我们可以针对合适的相干态选择得到能够描述该动力学的有效时空 (不同于标准量子场论中“有效”指积分掉高能模式的用法，在 LQC 的有效时空描述中，我们保留了量子几何的普朗克尺度效应。在 LQC 的 $\bar{\mu}$ 方案中，由有效哈密顿量得到的微分方程并不是经典的，除非我们探测的是小时空曲率区域。)。乍一看，人们自然会疑惑，怎么能在连续有效时空描述中囊括量子几何带来的非微扰量子引力效应？推导出该描述的核心结论是：薛定谔方程本质上就是量子相空间中的哈密顿方程 [148,149]。尽管对该方法的详细讨论超出了本章的范围，我们仍会简要介绍其核心思想，更多细节请读者查阅文献 [148-153]。

Using geometrical formulation of quantum mechanics, one can view the Hilbert space as a quantum phase space with a symplectic form Ω_Q defined via the imaginary part of the Hermitian inner product. The symplectic form allows us to define Poisson brackets and Hamiltonian vector fields. Given a self-adjoint operator $\widehat{O}$, its Schrodinger vector field can be defined as $X_{\widehat{O}}(\psi) := -(i/\hbar)\widehat{O}\psi$. If $\widehat{O}$ is a Hamiltonian operator, this is a Schrodinger equation. It turns out that the expectation value $\langle\widehat{O}\rangle$ also generates Hamiltonian vector field which is exactly the same as the one generated by $X_{\widehat{O}}$. This implies that if one is interested in obtaining the dynamical evolution of a quantum system determined by a self-adjoint Hamiltonian, then that can be equivalently obtained from the Hamilton's equations using expectation values of the Hamiltonian operator. If one considers two arbitrary self-adjoint operators $\widehat{A}$ and $\widehat{B}$ with expectation values $\bar{A} := \langle\psi|\widehat{A}|\psi\rangle$ and $\bar{B} := \langle\psi|\widehat{B}|\psi\rangle$, then it is straightforward to show that the Poisson bracket of $\bar{A}$ and $\bar{B}$ equals $\{\bar{A},\bar{B}\}_\Omega = \left\langle\frac{1}{i\hbar}\left[\widehat{A},\widehat{B}\right]\right\rangle$. Using which one can show that $\frac{\mathrm{d}}{\mathrm{d}t}\langle\widehat{A}\rangle = \{\bar{A},\bar{H}\}_{\Omega_Q}$. Time evolution of the expectation values of operator $\widehat{A}$ can be deduced from the Poisson bracket on the quantum phase space. It is to be noted that $\bar{H}$ is not a classical entity but is the expectation value of the quantum Hamiltonian operator. It is interesting that one can view the Schrödinger evolution as resulting from a Hamilton's equation in the quantum phase space.

利用量子力学的几何表述，我们可以将希尔伯特空间看作一个带有辛形式 Ω_Q 的量子相空间，该辛形式由厄米内积的虚部定义。借助辛形式我们可以定义泊松括号与哈密顿向量场。对于任意自伴算符 $\widehat{O}$，其薛定谔向量场可以定义为 $X_{\widehat{O}}(\psi) := -(i/\hbar)\widehat{O}\psi$。若 $\widehat{O}$ 是哈密顿算符，该向量场对应的就是薛定谔方程。可以证明，期望值 $\langle\widehat{O}\rangle$ 生成的哈密顿向量场与 $X_{\widehat{O}}$ 生成的完全一致。这意味着，如果我们想要得到自伴哈密顿量决定的量子系统的动力学演化，完全可以等价地通过哈密顿方程，利用哈密顿算符的期望值得到。若考虑任意两个自伴算符 $\widehat{A}$ 和 $\widehat{B}$，它们的期望值分别为 $\bar{A} := \langle\psi|\widehat{A}|\psi\rangle$ 和 $\bar{B} := \langle\psi|\widehat{B}|\psi\rangle$，不难证明 $\bar{A}$ 和 $\bar{B}$ 的泊松括号等于 $\{\bar{A},\bar{B}\}_\Omega = \left\langle\frac{1}{i\hbar}\left[\widehat{A},\widehat{B}\right]\right\rangle$，由此可以进一步得到 $\frac{\mathrm{d}}{\mathrm{d}t}\langle\widehat{A}\rangle = \{\bar{A},\bar{H}\}_{\Omega_Q}$。算符 $\widehat{A}$ 期望值的时间演化可以通过量子相空间上的泊松括号推导得到。需要注意的是，$\bar{H}$ 并非经典量，它是量子哈密顿算符的期望值。有意思的是，我们可以将薛定谔演化看作量子相空间中哈密顿方程导出的结果。

In LQC the effective spacetime description can be obtained by implementing above techniques and then obtaining an effective Hamiltonian. There are two ways this can be achieved. The first method is based on introducing a coordinate system on the infinite-dimensional quantum phase space using expectation values of quantum operators for phase space variables and their products and higher-order moments. Since dynamics is encoded in an infinite number of coupled nonlinear differential equations, one uses a truncation to a certain order in moments to obtain an approximate evolution of the expectation values [154]. The second method [21,22], which has been widely tested using numerical simulations [13,134,135], is based on finding a reliable embedding of the classical phase space into the quantum phase space. Finding this embedding which is preserved by the Hamiltonian flow is a nontrivial task which requires a judicious choice of semiclassical states. In LQC this embedding has so far been found for the spatially flat isotropic universe sourced with a massless scalar field. In the following we consider the effective Hamiltonian for the massless scalar case and generalize the setting to arbitrary matter assuming the validity of the effective Hamiltonian approach.

在圈量子宇宙学 (LQC) 中，可通过应用上述方法得到有效哈密顿量，从而获得有效时空描述。实现这一目标有两种途径: 第一种方法是，针对无穷维量子相空间，利用相空间变量量子算符的期望值、它们的乘积以及高阶矩来引入坐标系。由于动力学由无穷多组耦合非线性微分方程刻画，我们会将矩截断到特定阶数，从而得到期望值的近似演化 [154]。第二种方法 [21,22] 已通过数值模拟得到广泛验证 [13,134,135]，该方法基于将经典相空间可靠嵌入量子相空间。找到这种能被哈密顿流保持的嵌入并非易事，需要明智地选择半经典态。在 LQC 中，目前仅针对无质量标量场驱动的空间平坦各向同性宇宙找到了这种嵌入。下文我们将讨论无质量标量场情形的有效哈密顿量，并在假设有效哈密顿方法成立的前提下，将框架推广到任意物质情形。

The Modified Friedmann and Raychaudhuri Equations in the $k = 0$ FLRW Universe

$k = 0$ FLRW 宇宙中的修正弗里德曼方程和雷查德利方程

In a spatially flat FLRW universe, the effective Hamiltonian constraint has been derived in the case of the massless scalar field using the embedding method. In terms of v and b variables introduced in section "Spatially Flat, Homogeneous, and Isotropic Spacetime: Classical Aspects", it turns out to be [22]

在空间平坦的 FLRW 宇宙中，利用嵌入方法已经推导出了无质量标量场情形下的有效哈密顿约束。借助章节“空间平坦、均匀各向同性时空: 经典层面”中引入的 v 和 b 变量，最终可得 [22]

$$\mathcal{H} = -\frac{3v\sin^2(\lambda b)}{8\pi G\gamma^2\lambda^2} + \frac{p_\phi^2}{2v}, \tag{44}$$

where the matter content is taken to be a massless scalar field. Note that the same Hamiltonian can also be obtained from the classical Hamiltonian constraint (17) by using the thumb rule $b^2 \to \sin^2(\lambda b)/\lambda^2$ which as a cautionary remark only holds for the spatially flat FLRW universe. The modified Friedmann and Raychaudhuri equations can be derived from the Hamilton's equations

其中物质组分取为无质量标量场。注意，该哈密顿也可以通过经验规则 $b^2 \to \sin^2(\lambda b)/\lambda^2$ 从经典哈密顿约束 (17) 得到，需要提醒的是，该规则仅适用于空间平坦的 FLRW 宇宙。修正的弗里德曼方程与瑞查得符里方程可以从哈密顿方程推导出

$$\dot{v} = \frac{3v}{2\lambda\gamma}\sin(2\lambda b), \quad \dot{b} = -\frac{3\sin^2(\lambda b)}{2\gamma\lambda^2} - \frac{2\pi G\gamma p_\phi^2}{v^2}. \tag{45}$$

Using the Hamiltonian constraint and the equation of motion for the volume, it is straightforward to obtain the modified Friedmann equation [10,155]

利用哈密顿约束和体积的运动方程，可以很容易得到修正的弗里德曼方程 [10,155]

$$H^2 = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_c}\right). \tag{46}$$

Here ρ as before denotes the energy density of the matter content, which in the present case is $\rho = p_\phi^2/\left(2v^2\right)$, and ρ_c is the maximum density equal to $\rho_c = 3/8\pi G\lambda^2\gamma^2 \approx 0.41$ for $\gamma = 0.2375$, which is determined by the black hole thermodynamics in LQG [156]. If we assume the validity of the effective Hamiltonian constraint for all the matter content, the quantum bounce takes place at the maximum energy density ρ_c where the Hubble rate vanishes. Before the bounce, there appears a contracting phase which also has the classical limit in the distant past when the energy density becomes much less than the Planck density. The evolution of the universe filled with a massless scalar field is symmetric with respect to the quantum bounce, and the Hubble rate is bounded throughout the evolution. Moreover, the magnitude of the Hubble rate attains its maximum $H^2_{\max} = 1/4\gamma^2\lambda^2$ at $\rho = \rho_c/2$. In the regime $\rho_c/2 < \rho \leq \rho_c$, the universe enters into a super-inflationary phase with $\dot{H} > 0$.

此处和之前一样，ρ 表示物质组分的能量密度，在本文讨论的情形中为 $\rho = p_\phi^2/\left(2v^2\right)$，而 ρ_c 是最大密度，对于 $\gamma = 0.2375$ 来说最大密度等于 $\rho_c = 3/8\pi G\lambda^2\gamma^2 \approx 0.41$，该值由 LQG 中的黑洞热力学确定 [156]。如果假设有效哈密顿约束对所有物质组分都成立，那么量子反弹会发生在 Hubble 率消失的最大能量密度 ρ_c 处。反弹之前存在一个收缩相，当能量密度远低于普朗克密度时，该收缩相在极远过去也具有经典极限。填充无质量标量场的宇宙演化关于量子反弹对称，且 Hubble 率在整个演化过程中都是有界的。此外，Hubble 率的绝对值在 $\rho = \rho_c/2$ 处达到最大值 $H^2_{\max} = 1/4\gamma^2\lambda^2$。在 $\rho_c/2 < \rho \leq \rho_c$ 区域，宇宙进入满足 $\dot{H} > 0$ 的超暴胀相。

The Raychaudhuri equation can be similarly obtained from the equation of motion for b, which turns out to be

类似地，瑞查得符里方程可以从 b 的运动方程得到，最终结果为

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho\left(1 - 4\frac{\rho}{\rho_c}\right) - 4\pi G P\left(1 - 2\frac{\rho}{\rho_c}\right), \tag{47}$$

where P denotes the pressure of the scalar field. It is obvious that when the energy density is far less than the Planck density, both the modified Friedmann and Ray-chaudhuri equations asymptote to their classical counterparts. Finally, combining the modified Friedmann and Raychaudhuri equations, it is straightforward to obtain the energy conservation law

其中 P 表示标量场的压强。显然，当能量密度远小于普朗克密度时，修正后的弗里德曼方程和瑞查得符里方程都会渐近趋近于它们的经典形式。最后，结合修正的弗里德曼方程和瑞查得符里方程，可以很容易得到能量守恒定律

$$\dot{\rho} + 3H\left(\rho + P\right) = 0, \tag{48}$$

which takes the same form as its classical counterpart. The validity of these effective equations has been tested rigorously using high-performance computing methods [13, 134, 135], including extension to anisotropic models [14, 157]. It has been found that if one starts with states sharply peaked on the classical trajectory in the expanding branch and evolves them backward, then states retain their peakedness properties throughout the evolution and one recovers a classical pre-bounce universe before the bounce. If instead one considers a very quantum state, then the state retains its quantum character during the evolution through the bounce. Further, the energy density at bounce in such a case decreases [135] by exactly the same amount as predicted by exactly solvable model of spatially flat universe in LQC sourced with a massless scalar field [139]. For this particular model, bounce at lower density due to quantum fluctuations can also be captured by a modified Friedmann equation resembling (46) [158].

该定律形式与经典形式完全一致。这些有效方程的有效性已经通过高性能计算方法得到了严格验证 [13, 134, 135]，相关验证还推广到了各向异性模型 [14, 157]。研究发现，如果从膨胀分支经典轨道上尖峰分布的初态出发向后演化，态在整个演化过程中都会保持尖峰分布性质，我们可以在反弹前得到一个经典的反弹前宇宙。如果考虑的是一个高度量子化的态，那么该态在穿过反弹的演化过程中会保持其量子性质。此外，这种情形下反弹处的能量密度会降低 [135]，降低幅度恰好与填充无质量标量场的空间平坦宇宙 LQC 严格可解模型的预言一致 [139]。对于该特殊模型，量子涨落导致的低密度反弹也可以用类似 (46) 形式的修正弗里德曼方程描述 [158]。

We conclude this subsection with a few remarks. First let us note that the modifications to the Friedmann dynamics are non-perturbative in nature and cannot be captured by introducing a finite number of higher curvature terms. Indeed these arising from nonlocal effects require a nonlocal action. Such an effective action has been obtained in the Palatini framework which very accurately results in the modified Friedmann equation (46) [159]. Similarly an effective action has been obtained for the variants of LQC which result in much more complex modified Friedmann dynamics using Palatini framework [160]. Second, in the above treatment we have excluded inverse volume modifications in the effective dynamics. While they are not applicable for non-compact models, they still play a negligible role compared to holonomy modifications for compact spatially flat models if bounce happens at volumes much greater than Planck volume [10]. However, inverse volume modifications can play a role in singularity resolution in the presence of spatial curvature. In fact, they can lead to singularity resolution just by themselves [161], allow probing the nature of non-singular quantum spacetime near and at the zero scale factor [162], and are also important for bounds on anisotropic shear in Bianchi-IX model [163]. However, there is a caveat to be noted in the effective description. In the regime when inverse scale factor becomes important, fluctuations of states are expected to be large. But, as mentioned above, for states with large fluctuations, modified Friedmann equation surprisingly turns out to be valid but with a lower bounce density. While this result has so far been found for spatially flat model, note that for small-scale factors spatial curvature term is much smaller than the matter-energy density. While this does not provide any direct evidence of the validity of modified Friedmann like equations in the deep Planck regime for more general cases than as discussed in Ref. [158], it does indicate the domain of validity of modified Friedmann dynamics may be larger than expected. Nevertheless, in the following, it is important to recall the caveat of assuming the validity of effective dynamics in the entire regime.

我们在本小节最后做几点说明。首先需要指出，弗里德曼动力学的修改在本质上是非微扰的，无法通过引入有限数量的高阶曲率项得到。实际上，这些由非局域效应产生的修正需要非局域作用量描述。这类有效作用量已在帕拉蒂尼框架中得到，它能非常精确地给出修正后的弗里德曼方程(46)[159]。类似地，人们也在帕拉蒂尼框架中得到了 LQC 不同变体对应的有效作用量，这些变体给出了复杂得多的修正弗里德曼动力学 [160]。其次，在上述处理中我们排除了有效动力学中的逆体积修正。虽然逆体积修正不适用于非紧致模型，但对于紧致空间平坦模型，若反弹发生在远大于普朗克体积的尺度上，和环绕修正相比，逆体积修正的作用仍然可以忽略 [10]。不过，在存在空间曲率的情况下，逆体积修正可以在奇点解决中发挥作用。事实上，仅靠逆体积修正本身就可以实现奇点解决 [161]，它允许我们探究零尺度因子附近以及零尺度因子处非奇异量子时空的性质 [162]，并且对 Bianchi-IX 模型各向异性剪切的边界也十分重要 [163]。但需要注意有效描述存在一处需要说明的局限: 当逆尺度因子变得重要时，预计量子态的涨落会很大。不过如前文所述，对于涨落较大的态，修正后的弗里德曼方程出人意料地仍然成立，只是反弹密度更低。虽然这一结论目前仅在空间平坦模型中得到，但需要注意，当尺度因子很小时，空间曲率项远小于物质能量密度。这虽然不能为修正弗里德曼类方程在比文献 [158] 讨论的更一般情况的深普朗克区域中的有效性提供直接证据，但它表明修正弗里德曼动力学的有效范围可能比我们预期的更大。尽管如此，在下文中，我们必须牢记: 我们假设有效动力学在全部区域都成立，这是一个需要说明的局限。

Generic Resolution of Singularities

奇点的泛型消解

We have seen so far that quantum geometry effects in LQC can successfully resolve the big-bang singularity for isotropic universe sourced with a massless scalar field. If we assume that the effective Hamiltonian is valid for different types of matter in all regimes, it is easily seen that the modified Friedmann dynamics results in a bounce in the Planck regime if the matter satisfies the weak energy condition. However, as discussed in section ”Nature of Classical Singularities: Types, Strength, and Shapes”, cosmological singularities can be more general than the big-bang/crunch singularities. This results in a pertinent question: Whether loop quantum effects resolve all of the space-like singularities? To answer this question it is important to understand whether loop quantum geometric effects always bound the spacetime curvature, or if there are exceptions. What happens to the fate of strong and weak singularities? Finally, one would like to answer one of the most important questions on singularity resolution: Are loop quantum spacetimes geodesically complete?

我们此前已经看到，圈量子宇宙学 (LQC) 中的量子几何效应可以成功消解无质量标量场源各向同性宇宙的大爆炸奇点。如果我们假设有效哈密顿量对所有区域的不同类型物质都成立，不难发现: 只要物质满足弱能量条件，修正后的弗里德曼动力学就会在普朗克区域产生一次反弹。然而，正如“经典奇点的性质: 类型、强度与形态”一节所讨论的，宇宙学奇点可以比大爆炸/大挤压奇点更具一般性，这就引出了一个关键问题: 圈量子效应能否消解所有类空奇点？要回答这个问题，我们必须先弄清圈量子几何效应是否总能给时空曲率设置上限，还是存在例外。强奇点与弱奇点的最终命运是什么？最后，人们想要解答奇点消解领域最重要的一个问题: 圈量子时空是否测地完备？

While the geodesic extendibility tells us whether the event at which curvature diverges is a physical singularity or not, the strength of the singularities as discussed in section ”Types and Strength of the Singularities” tells us whether an in-falling observer or detector is completely annihilated by the tidal forces. These complimentary approaches together help us understand a more complete picture of physics of singularities. In classical cosmological context, geodesic incompleteness signals strong singularities and vice versa. Similarly, in the classical theory a passable curvature divergent event in geodesic evolution in a cosmological model is linked to a weak singularity. Note that quantum geometric effects can in principle resolve the strong singularity, but for a non-singular description it suffices if they just convert it into a weak one (It is possible that such a passage from a curvature divergent event in the effective description may result in additional complexities for the effective description. However, as everywhere else in this chapter we assume the validity of effective description in all regimes.).

> 测地可延拓性告诉我们曲率发散的事件是否属于物理奇点，而“奇点的类型与强度”一节讨论的奇点强度则告诉我们，下落的观测者或探测器是否会被潮汐力完全摧毁。这两种互补方法共同帮助我们建立对奇点物理更完整的认知。在经典宇宙学语境下，测地不完备性预示着强奇点，反之亦然。类似地，在经典理论中，宇宙学模型测地演化中可通过的曲率发散事件与弱奇点相关。需要注意的是，量子几何效应原则上可以消解强奇点，但要得到非奇异描述，只需将强奇点转化为弱奇点即可(这种从有效描述中曲率发散事件出发的转变，可能会给有效描述带来额外的复杂性，但和本章所有其他内容一样，我们假设有效描述在所有区域都成立)。

The fate of geodesic completeness and of strong and weak singularities has been explored in detail in LQC using the effective spacetime description [25]. Let us recall that using an exactly solvable model of LQC one can show that there is a nonzero lower bound on the expectation value of the volume operator and a universal upper bound on expectation value of the energy density operator for the states in the physical Hilbert space [15]. At the level of effective spacetime description, these results generalize to all types of matter but do not exclude the possibility of divergence in curvature components due to divergence in pressure at a finite density and scale factor. This is in addition to the boundedness of the Hubble rate which is directly responsible for lack of breakdown of geodesic evolution and resolution of strong singularities. The quantum geometric effects which manifest themselves in boundedness of relevant observables in the quantum theory, such as volume and energy density, result in an effective spacetime which is geodesically complete and free of strong singularities. It has been proved that for arbitrary matter content there can be no strong curvature singularities in the isotropic models [23] as well as in the presence of spatial curvature [122] and anisotropies which include Bianchi-I [24], Bianchi-II [26], and Bianchi-IX models [27]. These results also extend to the Kantowski-Sachs spacetime which in the absence of matter captures the interior of the Schwarzschild black hole [28]. In all these spacetimes, LQC results in geodesic completeness. In addition these results have been checked to be robust in modified versions of LQC for the spatially flat isotropic model [76]. An interesting result from these investigations is that LQC permits weak curvature singularities.

人们已利用有效时空描述，在 LQC 中详细探究了测地完备性以及强、弱奇点的结局 [25]。我们回顾: 利用 LQC 的精确可解模型可以证明，物理希尔伯特空间中，体积算符期望值存在非零下界，能量密度算符期望值存在普适上界 [15]。在有效时空描述层面，这些结论可推广至所有类型的物质，但并不排除在有限密度和标度因子下，压强发散导致曲率分量发散的可能性。而哈勃率有界这一点，直接导致测地演化不会中断，强奇点得以消解。量子几何效应体现为量子理论中体积、能量密度等相关可观测量有界，它带来的有效时空是测地完备且无强奇点的。已经证明，对于任意物质内容，各向同性模型 [23]、存在空间曲率的情况 [122]，以及包含 Bianchi-I[24]、Bianchi-II[26] 和 Bianchi-IX 模型 [27] 在内的各向异性情况中，都不存在强曲率奇点。这些结论也适用于 Kantowski-Sachs 时空——无物质时该时空描述史瓦西黑洞的内部 [28]。在所有这些时空中，LQC 都可实现测地完备。此外，人们已经验证，这些结论在空间平直各向同性模型的修正版本 LQC 中依然成立 [76]。这些研究得到一个有趣的结论:LQC 允许弱曲率奇点存在。

At these singularities spacetime curvature in LQC can indeed diverge. This can be easily seen via the modified Raychaudhuri equation (47). While the quantum geometry effects in LQC universally bound the energy density, they do not bind the pressure. Therefore, for an appropriate choice of equation of state where the pressure can diverge at finite energy density, the spacetime curvature can diverge in LQC. Recalling the types of singularities from section ”Types and Strength of the Singularities”, one finds that type-II singularities are not resolved in LQC [23]. Similarly, type-IV singularities which occur because of divergence in derivatives of spacetime curvature are also not resolved. Note that both of these singularities are weak singularities and thus harmless for geodesic evolution. In addition to these results, various models have been explored assuming phenomenological equation of state which permits study of various types of cosmological singularities [164- 168], in which the fate of various types of singularities has been studied. These investigations confirm that all types of strong curvature singularities are resolved in LQC. Indeed it can happen that strong curvature singularities are converted into weak ones because of quantum geometry effects making them harmless [23]. An open question in this arena is whether singularity resolution which is robustly seen for isotropic and anisotropic models also applies in the the presence of inhomogeneities.

在这些奇点处，圈量子宇宙学 (LQC) 中的时空曲率确实可以发散。这一点可以通过修正后的瑞查得符里方程 (47) 轻易看出。尽管 LQC 中的量子几何效应普遍给能量密度设置了上限，但它们并未限制压强。因此，当我们选取合适的物态方程，使得压强可以在有限能量密度下发散时，LQC 中时空曲率就可以发散。回顾“奇点的类型与强度”一节中对奇点的分类，我们可以发现 II 型奇点在 LQC 中并未得到解决 [23]。类似地，因时空曲率导数发散产生的 IV 型奇点同样未得到解决。需要注意的是，这两类奇点均为弱奇点，因此不会对测地线演化造成影响。除上述结论外，已有诸多研究基于唯象物态方程探索了各类模型，得以研究不同类型的宇宙奇点 [164- 168]，这些工作已经探究了各类奇点的最终演化结果。这些研究证实，所有类型的强曲率奇点在 LQC 中都能得到解决。实际上，量子几何效应确实会将强曲率奇点转化为弱奇点，使其不再有害 [23]。该领域目前有一个开放性问题: 在各向同性和各向异性模型中被稳定验证的奇点解决结论，是否同样适用于存在非均匀性的情况。

The Inflationary Scenario in LQC

LQC 中的暴胀情景

The inflationary paradigm is one of the cornerstones in the modern cosmology. It does not only resolve some long-standing puzzles in the standard big-bang model but also provides a mechanism for the development of large-scale structure in the universe. However, the inflationary paradigm itself is not past complete as pointed out in the incompleteness theorem by Borde, Guth, and Vilenkin [103], and the inflationary spacetime in a spatially flat FLRW universe would inevitably encounter a singularity in the backward evolution as long as the null energy condition is satisfied [3]. To resolve the big-bang singularity in the inflationary paradigm, a straightforward phenomenological model in LQC is to make use of the effective dynamics by adding an inflationary potential into the effective Hamiltonian constraint. Although with the inclusion of the inflationary potential, the scalar field can no longer serve as a matter clock due to its non-monotonicity, the additional matter clocks (also called reference fields), such as the dust fields or the massless Klein-Gordon fields, can be taken into account, resulting in a ”two-fluid” system [96]. The loop quantization of such a system is implemented in the reduced phase space where the Dirac observables with respect to the matter clock are explicitly constructed. The quantization proceeds in a similar way with the $\bar{\mu}$ scheme as described in section ”Loop Quantum Cosmology: Spatially Flat Isotropic Model”, leading to the quantum difference equations with the same non-singular structure as in LQC. It turns out that the effects of the reference fields on the background evolution of the universe can be tuned as small as possible when the magnitudes of their energy densities are chosen to be sufficiently small [96]. Therefore, in the following, we simply add the inflationary potential directly to the effective Hamiltonian constraint as a phenomenological model to consider the effects of the inflationary potential in an LQC universe without taking into account any additional effects from the reference fields. This is also the common practice in the literature to study the extension of the inflationary paradigm to the Planck regime in the framework of LQC. Moreover, numerous inflationary models have been discussed extensively in LQC, such as single-field inflation [29, 31-34, 39-45], multi-field inflation [35], tachyonic inflation [30], inflation with non-minimally coupled scalar field [36], in the presence of anisotropies [37], spatial curvature [46, 169, 170], and warm inflationary scenarios which include radiation dissipation [47].

暴胀范式是现代宇宙学的基石之一。它不仅解决了标准大爆炸模型中一些长期存在的难题，还为宇宙大尺度结构的形成提供了机制。然而，正如 Borde、Guth 和 Vilenkin 在不完备性定理中指出的 [103]，暴胀范式本身并非过去完备；只要满足零能量条件 [3]，空间平坦 FLRW 宇宙中的暴胀时空在反向演化中不可避免地会遇到奇点。为解决暴胀范式中的大爆炸奇点，圈量子宇宙学 (LQC) 中一个直接的唯象模型是利用有效动力学，将暴胀势引入有效哈密顿约束中。尽管引入暴胀势后，标量场因非单调性无法再用作物质时钟，但我们可以引入额外的物质时钟 (也称为参考场)，例如尘埃场或无质量克莱因-戈登场，从而形成” 双流体” 系统 [96]。该系统的圈量子化在约化相空间中完成，其中已明确构造了相对于物质时钟的狄拉克可观测量。量子化过程与” 圈量子宇宙学: 空间平坦各向同性模型” 一节描述的 $\bar{\mu}$ 方案类似，最终得到的量子差分方程具有与 LQC 中相同的非奇异结构。研究表明，只要将参考场的能量密度取值足够小，参考场对宇宙背景演化的影响就可以调控到任意小 [96]。因此在下文中，我们作为唯象模型直接将暴胀势加入有效哈密顿约束，研究 LQC 宇宙中暴胀势的效应，不考虑参考场带来的额外影响，这也是文献中在 LQC 框架下研究暴胀范式向普朗克能标延伸的常用做法。此外，LQC 中已经对大量暴胀模型进行了广泛讨论，包括单场暴胀 [29, 31-34, 39-45]、多场暴胀 [35]、快子暴胀 [30]、非最小耦合标量场暴胀 [36]、存在各向异性的暴胀 [37]、存在空间曲率的暴胀 [46, 169, 170]， 以及包含辐射耗散的暖暴胀情景 [47]。

Since Planck 2018 data prefers single-field inflationary model with a plateau in a spatially flat universe [171], we will focus on this simplest model of inflation in the rest of this subsection. The effective Hamiltonian for a massive scalar field minimally coupled to gravity in LQC takes the form

由于 2018 年普朗克数据更支持空间平坦宇宙中的单场平台型暴胀模型 [171]，我们将在本小节剩余部分聚焦于这个最简单的暴胀模型。LQC 中最小耦合引力的有质量标量场的有效哈密顿量形式为

$$\mathcal{H} = -\frac{3v\sin^2(\lambda b)}{8\pi G\gamma^2\lambda^2} + \frac{p_\phi^2}{2v} + vU, \tag{49}$$

where U stands for the inflationary potential. Using the effective dynamics generated by the above background Hamiltonian constraint, various inflationary potentials with their phenomenological implications on the background evolution of the LQC universe have been investigated in the last decade, such as the chaotic potential, the fractional monodromy potential, the Starobinsky potential, the non-minimal Higgs potential, and the α attractor. Among them, the most studied examples are the chaotic potential and the Starobinsky potential. It has been found that although the bounce can be dominated by either the kinetic energy or the potential energy of the inflaton field, only the kinetic-energy-dominated bounce turns out to be relevant to the observations, which requires around 60 inflationary e-foldings. The potential-energy-dominated bounce can exhibit distinct properties of the inflationary phase with a different choice of the potential. For example, with the chaotic potential, the potential-energy-dominated bounce can lead to excessively long period of the inflationary phase, while with the Starobinsky potential, there would be even no sustained period of the inflationary phase if the bounce is dominated by the potential energy. As a result, various investigations have been directed to study the properties of the universe with a kinetic-energy-dominated bounce by using numerical simulations as well as the analytical approximations. It has been found that such a universe undergoes several characteristic stages evolving from the quantum bounce in the Planck regime [43]. These stages are featured by the distinct behavior of the equation of state of the inflaton field. The bounce, including the super-inflationary phase, is located in the Planck regime where the equation of state is very close to unity due to the negligible potential energy. It is then followed by the transition phase in which the equation of state quickly drops from positive to negative unity in a couple of e-foldings. Afterward, the slow-roll phase starts, and its duration is directly determined by the initial value of the inflaton field at the quantum bounce. In addition to the numerical results, the analytic expressions of the background solutions have also been studied for each phase particularly for the chaotic and the Starobinsky potentials which serve as the basis for an analytical investigation of the primordial power spectra in LQC later on [43].

其中 U 为暴胀势。利用上述背景哈密顿约束给出的有效动力学，过去十年研究者已经对各类暴胀势及其对 LQC 宇宙背景演化的唯象影响开展了研究，包括混沌势、分数单环绕势、斯塔罗宾斯基势、非最小希格斯势以及 α 吸引子。其中研究最多的是混沌势和斯塔罗宾斯基势。研究发现，尽管量子反弹既可以由暴胀场的动能主导，也可以由势能主导，但只有动能主导的反弹才符合观测结果，这类反弹要求暴胀产生约 60 个 e 折叠。势能主导反弹的暴胀阶段性质会随势的选取不同而变化: 例如选取混沌势时，势能主导反弹会导致暴胀阶段过长；而若选取斯塔罗宾斯基势，势能主导反弹甚至无法产生持续的暴胀阶段。因此，现有研究大多通过数值模拟和解析近似研究动能主导反弹下的宇宙性质。研究发现，这类宇宙从普朗克能标的量子反弹开始演化，会经历多个特征阶段 [43]，不同阶段以暴胀场物态方程的不同行为为区分。包含超暴胀阶段在内的反弹过程位于普朗克能标，此阶段势能可忽略，物态方程非常接近 1；随后进入过渡阶段，物态方程在数个 e 折叠内从正值快速降至-1；之后慢滚阶段开始，该阶段的持续时间直接由量子反弹处暴胀场的初始值决定。除数值结果外，研究者还得到了各阶段背景解的解析表达式，其中混沌势和斯塔罗宾斯基势的结果是后续 LQC 原初功率谱解析研究的基础 [43]。

On the other hand, to fully explore the parameter space of the initial conditions at the quantum bounce which is essentially one-parameter space determined by the value of the inflaton field, the qualitative dynamics of the inflationary model in LQC is analyzed by casting the universe filled with an inflaton field into an autonomous system described by a closed set of first-order differential equations [29]. The two-dimensional phase space portraits can be obtained from this set of equations with x/y-axis proportional to the square root of the potential/kinetic energy of the inflaton field. A representative example of the phase space portraits is illustrated in Fig. 2 in which we use the Starobinsky potential as an example. All the trajectories are confined within the unit circle which represents the quantum bounce at the maximum energy density in LQC. From the above phase space portraits, one can intuitively see that ”most” of the initial conditions can lead to the slow-roll phase before reaching the reheating phase located at the center of the plot. In LQC, one can unambiguously compute the probability for the occurrence of the desirable slow-roll phase which is defined as the one with enough e-foldings to allow the pivot mode to exit the Hubble horizon during the slow-roll phase. The key point is the additional structure provided by the quantum bounce in LQC. As is well known, the problem with having a definite probability for the occurrence of the desirable slow-roll phase in classical GR is tied up to the fact that the Liouville measure on the gauge-fixed surfaces $H = H_0 =$ const fails to naturally descend to a measure in the space of physically distinct solutions. As a result, different choices of the gauge-fixed surface, namely different values of H_0 , can even lead to opposite results on the probability [172]. In contrast, there exists a canonical choice of the time instant, i.e., the bounce surface, which is endowed with a distinguished physical measure yielding a finite physical volume of the available parameter space at the bounce. It can be shown that for the chaotic potential, the possibility for the desirable slow-roll phase to not happen in LQC is less than three in a million [32,34,38]. Similar results have been obtained for other versions of LQC obtained from treating Euclidean and Lorentzian parts of the Hamiltonian constraint independently resulting in mLQC-I and mLQC-II [173, 174]. No such are available for the Starobinsky potential. This is because the physical volume of the parameter space turns out to be infinite, and hence no estimates on the possibility for the desirable slow-roll to happen can be found.

另一方面，为充分探索量子反弹处初始条件的参数空间——该空间本质是由暴胀场取值决定的单参数空间，研究者将充满暴胀场的 LQC 宇宙转化为由一阶封闭微分方程组描述的自治系统，分析了暴胀模型的定性动力学 [29]。由此可得到二维相空间图，其横/纵坐标分别正比于暴胀场势能/动能的平方根。图 2 给出了以斯塔罗宾斯基势为例的典型相空间图: 所有轨迹都被限制在单位圆内，该圆对应 LQC 最大能量密度处的量子反弹；从相空间图可以直观看出，“绝大多数”初始条件都会在到达图中心的再加热阶段之前进入慢滚阶段。在 LQC 中，我们可以明确计算出合格慢滚阶段 (即拥有足够 e 折叠数、使得 pivot 模式在慢滚阶段退出哈勃视界的慢滚阶段) 发生的概率，关键在于 LQC 中量子反弹提供了额外结构。众所周知，经典广义相对论中无法确定合格慢滚阶段发生的概率，根源在于规范固定面 $H = H_0 = =$const 上的刘维尔测度无法自然导出物理 distinct 解空间上的测度，因此选取不同的规范固定面 (即 H_0 取不同值) 甚至会得到完全相反的概率结果 [172]。与此不同，LQC 中存在规范时刻的典范选取，即反弹面，该面具有唯一的物理测度，可得到反弹处可用参数空间的有限物理体积。可以证明，对于混沌势，LQC 中不发生合格慢滚阶段的概率小于百万分之三 [32,34,38]；对分别独立处理哈密顿约束欧几里得部分和洛伦兹部分得到的其他 LQC 版本 (即 mLQC-I 和 mLQC-II)，也得到了类似结论 [173, 174]。但斯塔罗宾斯基势尚无这类结果，因为其参数空间的物理体积是无穷大，因此无法估计合格慢滚阶段发生的概率。

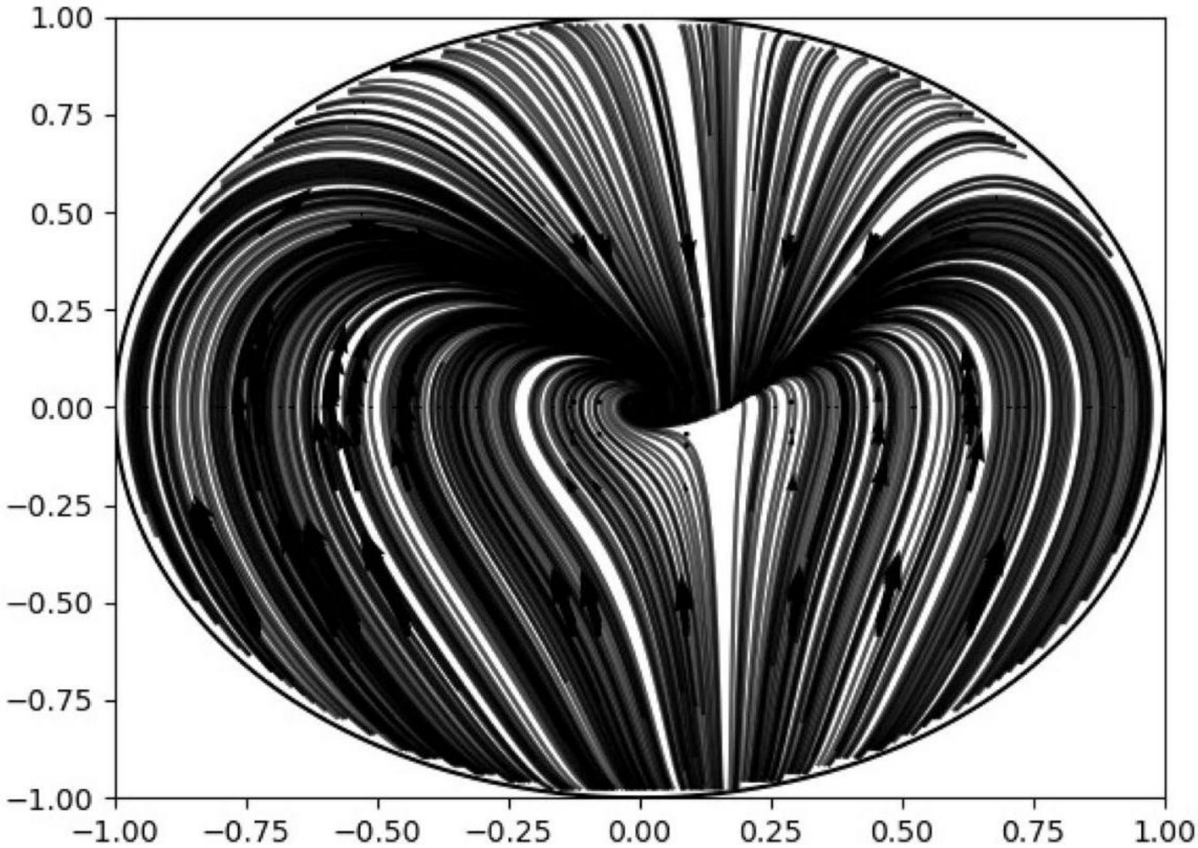

Fig. 2 This figure shows a 2-dimensional phase space portrait of the qualitative dynamics of the universe filled with a massive scalar field with mass m for the Starobinsky potential in LQC. The x -axis is $X = \chi_0\left(1 - \exp\left(-\sqrt{16\pi G/3}\phi\right)\right)$ with $\chi_0 = \sqrt{\frac{3m^2}{32\pi G\rho_c}}$. The y-axis is $Y = \dot{\phi}/\sqrt{2\rho_c}$. It illustrates the generic evolution of the physical solutions from the quantum bounce to the reheating phase. The arrows in the figure denote the time flow in the forward evolution. All the trajectories starting from the unit circle which represents the quantum bounce first tend to the inflationary separatrices and then circling around the origin, signifying the decaying of the energy density in the reheating phase

图 2 本图展示了循环量子宇宙学中，含质量 m 的标量场、斯塔罗宾斯基势的宇宙定性动力学的二维相空间图。x 轴为 x ，对应 $X = \chi_0\left(1 - \exp\left(-\sqrt{16\pi G/3}\phi\right)\right)$ 和 $\chi_0 = \sqrt{\frac{3m^2}{32\pi G\rho_c}}$ 。y 轴为 $Y = \dot{\phi}/\sqrt{2\rho_c}$ 。它展示了物理解从量子反弹到再加热阶段的一般演化过程。图中箭头表示正向演化的时间流向。所有从代表量子反弹的单位圆出发的轨迹，先趋向暴胀分界线，随后绕原点环绕，象征再加热阶段能量密度衰减

The Matter-Bounce and the Ekpyrotic Scenarios in LQC

圈量子宇宙学中的物质反弹与火劫场景

In this subsection, we briefly summarize the results from the ekpyrotic and the matter-ekpyrotic-bounce scenario in LQC. The matter-bounce scenario and the ekpyrotic scenario are actually two separate ansatz to address different issues encountered in the construction of alternative phenomenological models to the inflationary paradigm (Note that as in the case of inflationary potentials studied in LQC, the ekpyrotic potential is not derived from LQG and is used in the following discussion only as a phenomenological model to understand the role of quantum geometric effects in the presence of ekpyrosis which here means a period where equation of state is greater than unity.). The matter-bounce scenario [175] aims to produce a scale-invariant power spectrum of the linear perturbations during the contracting phase of the universe in a period of vanishing equation of state of the matter field which is dual to the inflationary phase in the expanding branch [176]. In GR, the most difficult part for such a scenario is the realization of a nonsingular bounce, which is usually achieved by introducing an exotic matter content which violates null energy condition or modified gravitational sector [177-181]. In contrast, the generic resolution of the big-bang singularity due to the quantum geometry effects in LQC provides a natural framework to investigate the physical implications of such a scenario without altering either the geometric or the matter sectors. There is no need to introduce any exotic matter or fine-tuning of initial conditions, and one can simply consider a massless scalar field or a dust field in the bouncing universe in LQC. However, it turns out that the magnitude of the power spectra produced in such a simple setting is proportional to the maximum energy density in LQC which is of the Planck scale [50]. Hence, considering only a matter field with vanishing equation of state in LQC is not sufficient to produce a scale-invariant power spectrum with the observationally consistent magnitude.

在本小节中，我们简要总结圈量子宇宙学 (LQC) 中火劫场景与物质-火劫-反弹场景的研究结果。物质反弹场景和火劫场景实际上是两个独立的假设，用于解决 inflation 范式之外的替代唯象模型构建中遇到的不同问题 (注意，和 LQC 中研究的 inflation 势情况一样，火劫势并非来源于圈量子引力 (LQG)，以下讨论仅将其作为唯象模型，用于理解存在火劫过程时量子几何效应的作用——此处火劫过程指物态方程大于 1 的阶段。)。物质反弹场景 [175] 旨在宇宙收缩阶段，产生标度不变的线性扰动功率谱，该阶段物质场的物态方程为零，对应膨胀分支中 inflation 阶段的对偶 [176]。在广义相对论 (GR) 中，这类场景最难解决的问题是实现非奇异反弹，通常需要引入违反零能量条件的奇异物质，或是修改引力 sector [177-181] 才能实现。与之相对，LQC 中量子几何效应带来的大爆炸奇点通用解决，为研究这类场景的物理意义提供了自然框架，无需修改几何或物质 sector。不需要引入任何奇异物质，也不需要对初始条件进行精细调节，只需在 LQC 的反弹宇宙中考虑一个无质量标量场或尘埃场即可。但研究发现，在这种简单设置下产生的功率谱幅度与 LQC 中 (普朗克尺度的) 最大能量密度成正比 [50]。因此，仅在 LQC 中考虑物态方程为零的物质场，不足以产生幅度符合观测的标度不变功率谱。

On the other hand, the ekpyrotic scenario provides an alternative explanation on the origin of the universe [182, 183]. In this framework, our universe is supposed to exist on a brane interacting with other branes in a higher-dimensional bulk. The big-bang/big crunch singularities correspond to the collision of branes, and the universe is supposed to undergo cycles of contracting and expanding phases. This evolution is captured by a moduli field which measures the inter-brane distance. The inter-brane potential has a peculiar shape with a sharp negative well. When the moduli field is in this well, its equation of state can become much greater than unity. In this case ekpyrotic matter-energy density dominates the dynamics even in the presence of anisotropies which scales as $\sigma^2 \propto a^{-6}$. In principle this phenomena can alleviate the chaotic approach to singularities and the Belinski-Khalatnikov-Lifshitz (BKL) instability [184]. But the problem with conventional ekpyrotic scenarios is the presence of big-bang/big crunch singularity. In LQC, the ekpyrotic scenario was first discussed in [29, 48] where singularity resolution was achieved. It was noted that even with singularity resolution there exist inherent problems, which are independent of LQC, with lack of turnaround of the moduli field along with that of the scale factor at the bounce. Later it was found that in order to form a cyclic universe, in addition to the ekpyrotic field, another matter field or anisotropies must be also taken into account [49]. Extensive numerical simulations have been performed in the Bianchi-I model in LQC with ekpyrotic and an ekpyrotic-like potential which confirm resolution of singularity in these models due to non-perturbative effects [185].

> 另一方面，火劫场景对宇宙起源给出了另一种解释 [182, 183]。在该框架中，我们的宇宙被认为存在于高维 bulk 中与其他膜相互作用的膜上。大爆炸/大挤压奇点对应膜的碰撞，宇宙被认为经历收缩与膨胀的循环演化。这种演化由测量膜间距的模场描述。膜间势具有特殊的形状，带有一个陡峭的负势阱。当模场处于该势阱中时，它的物态方程可以远大于 1。此时，即便存在按 $\sigma^2 \propto a^{-6}$ 标度的各向异性，火劫物质能量密度仍主导动力学。原则上，这一现象可以缓解通往奇点的混沌趋近与别连斯基-哈拉特尼科夫-利夫希茨 (BKL) 不稳定性 [184]。但传统火劫场景的问题在于存在大爆炸/大挤压奇点。LQC 中最早在文献 [29, 48] 讨论了火劫场景，实现了奇点解决。研究指出，即便奇点得到解决，该场景仍存在与 LQC 无关的固有问题: 模场和标度因子在反弹处都无法转向。后来人们发现，要形成循环宇宙，除了火劫场之外，还必须考虑另一个物质场或各向异性 [49]。人们已经对 LQC 中的 Bianchi-I 模型做了大量数值模拟，引入火劫势和类火劫势，结果证实非微扰效应会解决这些模型中的奇点问题 [185]。

Since two scenarios discussed above address different problems encountered in the construction of a feasible alternative paradigm to the inflationary one, it is useful to study them together which results in the matter-ekpyrotic-bounce scenario [186, 187]. In LQC, the physical implications of the matter-ekpyrotic-bounce scenario on the background dynamics as well as the primordial power spectra have been investigated in [51-53]. In these works, gravity is minimally coupled to two fluids, namely a dust field which plays the role of producing the scale-invariant power spectra in the contracting phase and the ekpyrotic field which isotropizes the bounce. The contracting branch is thus divided into two stages, the dust-dominated phase far from the bounce and the ekpyrotic phase near the bounce which helps remove the reliance of the magnitude of the power spectra on the maximum energy density at the bounce in LQC. Using the dressed metric approach to perturbations, it has been found that the primordial power spectrum of co-moving curvature perturbations is almost scale invariant for the modes which exit the horizon in the matter-dominated phase [53]. Analysis of the spectral index and observational constraints shows that refinements in this model are necessary. Finally, let us note that while ekpyrotic models can be successfully embedded in LQC, the same cannot be said if we generalize to another variant of LQC [188] (see section ”The Effective Dynamics of mLQC-I/II” for a discussion).

> 由于上述两种情景针对构建可行的暴胀替代范式过程中遇到的不同问题, 因此将二者结合研究是有意义的, 由此得到了物质-火劫反弹情景 [186, 187]。在圈量子宇宙学 (LQC) 中，文献 [51-53] 已经研究了物质-火劫反弹情景对背景动力学和原初功率谱的物理影响。在这些研究中，引力与两种流体最小耦合: 一种是尘埃场，其作用是在收缩阶段产生标度不变的功率谱；另一种是火劫场，其作用是使反弹各向同性。因此，收缩分支被分为两个阶段: 远离反弹的尘埃主导阶段，以及靠近反弹的火劫阶段——火劫阶段有助于消除 LQC 中功率谱幅度对反弹处最大能量密度的依赖。利用微扰的修饰度规方法研究发现，对于在物质主导阶段出视界的模式，共动曲率扰动的原初功率谱几乎是标度不变的 [53]。对谱指数和观测限制的分析表明，该模型仍需要进一步优化。最后需要指出，尽管火劫模型可以成功嵌入 LQC，但如果推广到 LQC 的另一个变体，结论就不成立了 [188](相关讨论见章节“mLQC-I/II 的有效动力学”)。

Loop Quantization in the Presence of Anisotropies and Inhomogeneities

各向异性与非均匀性存在下的圈量子化

In this section, we overview the loop quantization of the Bianchi-I model in the $\overline{\mu}$ scheme and the polarized Gowdy models with the inhomogeneities as infinite degrees of freedom. When extending the techniques developed for the isotropic model to the cases in the presence of anisotropies such as Bianchi-I spacetime, one has to handle several new difficulties. These include defining a curvature operator in terms of the holonomies which depend on the directional triads in a nontrivial way, finding a proper set of the dynamical variables that can result in a manageable Hamiltonian constraint operator, etc. More importantly, the resulting Hamiltonian constraint operator should not suffer from the drawbacks encountered in the μ_0 scheme of the isotropic case as discussed earlier. It turns out that there exist two quantization prescriptions, one known as Madrid prescription [58,120] and another known as Ashtekar-Wilson Ewing prescription [59], which result in the $\overline{\mu}$ scheme in the isotropic limit. However, the former becomes unviable with $\mathbb{R}^3$ topology [124] and also suffers from problematic features at large volumes [189]. For the Ashtekar-Wilson prescription, cosmological implications on the non-singular evolution of the Bianchi-I universe can be found in [24, 37, 185, 190] . For the loop quantization of other anisotropic models using same prescription, see [163, 191, 192].

在本节中，我们概述 $\overline{\mu}$ 方案下 Bianchi-I 模型，以及将非均匀性作为无穷自由度的极化 Gowdy 模型的圈量子化。将各向同性模型发展出的技术推广到 Bianchi-I 时空这类存在各向异性的情况时，需要处理若干新难点。其中包括: 以和方向三元非平凡关联的环绕量定义曲率算符，找到能得到可处理的哈密顿约束算符的合适动力学变量集等。更重要的是，得到的哈密顿约束算符不能存在我们之前讨论过的各向同性情形 μ_0 方案中出现的缺陷。目前已存在两种量子化方案，一种名为马德里方案 [58,120]，另一种名为 Ashtekar-Wilson Ewing 方案 [59]，二者在各向同性极限下都会得到 $\overline{\mu}$ 方案。但前者在 $\mathbb{R}^3$ 拓扑下不成立 [124]，且在大体积处存在问题特征 [189]。对于 Ashtekar-Wilson 方案，可在 [24, 37, 185, 190] 中找到其对 Bianchi-I 宇宙非奇异演化的宇宙学相关结论。关于其他各向异性模型使用该方案的圈量子化研究，参见 [163, 191, 192]。

In addition to the spacetimes with anisotropies, it is also equally important to incorporate inhomogeneities as a step toward understanding the quantum geometry effects and investigating the robustness of the singularity resolution in the presence of the infinite degrees of freedom. This provides useful insights on the possible properties of the cosmological sector of the full theory. In the following, we will summarize the main results of the $\overline{\mu}$ scheme quantization of the Bianchi-I model and discuss its phenomenological implications by using the effective dynamics. Then we will move onto a brief summary of the simplest Gowdy T^3 which uses the construction of Bianchi-I spacetime.

除了存在各向异性的时空，引入非均匀性同样重要，这是一步必要的推进，可帮助我们理解量子几何效应，探究存在无穷自由度时奇点解决机制的鲁棒性，还能为我们提供全理论宇宙学部分可能性质的有用洞察。下文我们将总结 Bianchi-I 模型 $\overline{\mu}$ 方案量子化的主要结果，利用有效动力学讨论其唯象学推论，随后简要总结基于 Bianchi-I 时空构造的最简单 Gowdy T^3 的相关内容。

Loop Quantization of Bianchi-I Spacetime and its Effective Dynamics

比安基-I 时空的圈量子化及其有效动力学

A detailed construction of the quantum theory of loop quantized Bianchi-I spacetime is addressed in [59]. Here we outline the basic ideas and procedures of the quantization which is implemented in a similar way as that of the isotropic model discussed in section "Loop Quantization of the Spatially Flat FLRW Universe". The novel features of the quantization come from the presence of anisotropies which result in six degrees of freedom in the classical phase space, namely the directional connection variables c_i and the directional triads p_i . Correspondingly, the fundamental variables for the loop quantization are the directional triads and the holonomies of the directional connection variables. Using the same techniques applied in the isotropic case, one can construct the kinematic Hilbert space in the triad representation with the almost periodic functions of the connection as the basis states. Meanwhile, to construct a physically viable Hamiltonian constraint operator, one of the key elements is to find a proper regularization of the field strength operator. In the isotropic case, such an operator is defined on a square $\square_{ij}$ as computed exactly in (37). With the physical area of the square shrinking to the minimal nonzero eigenvalue of the area operator in the homogeneous spacetime, the unique $\bar{\mu}$ scheme is obtained. In the case of the Bianchi-I spacetime, the edge length $\bar{\mu}_i$ in each direction is determined from a correspondence between the kinematic states in LQG and those in LQC which yields the specification of the edge length as

文献 [59] 详细构造了圈量子化比安基-I 时空的量子理论。本文概述该量子化的基本思路与步骤，其实现方式与“空间平坦 FLRW 宇宙的圈量子化”一节讨论的各向同性模型类似。该量子化的新特征源于各向异性的存在，各向异性使得经典相空间存在六个自由度，即方向联络变量 c_i 和方向标架 p_i 。相应地，圈量子化的基本变量是方向标架和方向联络变量的完整绕。利用各向同性情形下的相同技术，可以在标架表象中构造运动学希尔伯特空间，以联络的概周期函数为基态。同时，构造物理上可行的哈密顿约束算符的关键步骤之一，是找到场强算符的合适正则化。在各向同性情形下，该算符定义在如式 (37) 精确计算的正方形 $\square_{ij}$ 上。随着正方形的物理面积收缩至均匀时空面积算符的最小非零本征值，我们得到了唯一的 $\bar{\mu}$ 方案。在比安基-I 时空情形下，每个方向的边长 $\bar{\mu}_i$ 由 LQG 运动学态与 LQC 运动学态之间的对应关系确定，该对应关系将边长规定为

$$\bar{\mu}_1 = \lambda\sqrt{\frac{|p_1|}{|p_2 p_3|}},\ \bar{\mu}_2 = \lambda\sqrt{\frac{|p_2|}{|p_3 p_1|}},\ \bar{\mu}_3 = \lambda\sqrt{\frac{|p_3|}{|p_1 p_2|}}. \tag{50}$$

When anisotropies disappear, the above value of the edge length in each direction reduces to the one used in the $\bar{\mu}$ scheme in the isotropic case given in (38). Another key observation in the quantization procedure is the use of a new set of dynamical variables which can lead to a manageable Hamiltonian constraint operator. It turns out that instead of using the states $\Psi(p_1, p_2, p_3)$, it is more convenient to work out the action of the Hamiltonian constraint operator on the wavefunction $\Psi(\lambda_1, \lambda_2, v)$ with

当各向异性消失时，上述各方向边长退化为式 (38) 给出的各向同性情形 $\bar{\mu}$ 方案中使用的边长。量子化过程的另一个关键结论是，使用一组新的动力学变量可以得到易于处理的哈密顿约束算符。研究发现，相比使用态 $\Psi(p_1, p_2, p_3)$ ，在波函数 $\Psi(\lambda_1, \lambda_2, v)$ 上推导哈密顿约束算符的作用更为简便，其中

$$\lambda_i = \frac{\operatorname{sgn}(p_i)\sqrt{|p_i|}}{\left(4\pi\gamma\lambda l_{\mathrm{Pl}}^2\right)^{1/3}}, \ \upsilon = 2\lambda_1\lambda_2\lambda_3, \tag{51}$$

where $i = 1, 2, 3$ and λ_i has the same sign as p_i . In the absence of the fermions, the physical wavefunction $\Psi(\lambda_1, \lambda_2, \upsilon)$ should be symmetric under a flip in the orientation of the fiducial triads in each direction, demanding the requirement $\Psi(\lambda_1, \lambda_2, \upsilon) = \Psi(|\lambda_1|, |\lambda_2|, |\upsilon|)$, and meanwhile be annihilated by the Hamiltonian constraint operator, leading to a dynamical difference equation whose evolution is again unfolded with respect to the massless scalar field. The exact form of the quantum difference equation is rather complicated as compared with its isotropic cousin given in (42). Therefore, we will not explicitly cite it here, and interested readers can refer to Eqs. (3.13)-(3.16) in [59]. The quantum difference equation in the $\bar{\mu}$ scheme for the Bianchi-I spacetime also has the uniform steps in the volume, that is, it involves only the physical states with the volumes υ and $\upsilon \pm 4$. For this reason, there also exist superselection sectors with support on lattices $\mathcal{L}_{\pm\varepsilon}$ as in the isotropic case. However, there is no superselection with respect to the first two arguments, namely λ_1 and λ_2 , of the wavefunction. Moreover, λ_1 and λ_2 , which carry the information of the anisotropies, do not appear in the coefficients of the quantum difference equation but only appear in the arguments of the wavefunction. Under the action of the Hamiltonian constraint operator they get rescaled by factors depending only on the volume. Owing to this property, it turns out that there exists a natural projection from the physical states in the Bianchi-I Hilbert space to those in the isotropic Hilbert space. When integrating out the anisotropic degrees of freedom, the quantum difference equation of the Bianchi-I model can exactly reduce to the one in the isotropic $k = 0$ FLRW model [59]. This provides a typical example of how the symmetry reduced quantum dynamics can be obtained from a more general one. Considering the BKL conjecture which asserts that the dynamics of GR near space-like singularities can be well modeled by Bianchi-I cosmology, the projection of the Bianchi-I model to the isotropic FLRW model implies that the latter may capture some genetic features of the quantum dynamics of the homogeneous and isotropic sector of the full theory. Finally, it can be shown that the quantum Bianchi-I difference equation can be well approximated by the Wheeler-DeWitt differential equation when the quantum geometry effects are negligible. However, in the Planck regime, this approximation fails and two quantum theories lead to distinct physical predictions as in the isotropic case.

其中 $i = 1, 2, 3$ 和 λ_i 与 p_i 符号相同。不存在费米子时，物理波函数 $\Psi(\lambda_1, \lambda_2, \upsilon)$ 在各方向基准三分量取向翻转下应对称，因此要求满足条件 $\Psi(\lambda_1, \lambda_2, \upsilon) = \Psi(|\lambda_1|, |\lambda_2|, |\upsilon|)$ ，同时它需被哈密顿约束算符零化，由此得到动力学差分方程，该方程的演化同样是针对无质量标量场展开的。与式 (42) 给出的各向同性对应形式相比，该量子差分方程的具体形式相当复杂，因此我们不在此明确列出，感兴趣的读者可查阅文献 [59] 中的式 (3.13)-(3.16)。Bianchi-I 时空在 $\bar{\mu}$ 方案下的量子差分方程同样在体积上具有均匀步长，即它仅涉及体积为 υ 和 $\upsilon \pm 4$ 的物理态。因此，和各向同性情况一样，该模型也存在支集在格点 $\mathcal{L}_{\pm\varepsilon}$ 上的超选择区。但波函数的前两个自变量，即 λ_1 和 λ_2 不存在超选择。此外，携带各向异性信息的 λ_1 和 λ_2 不出现在量子差分方程的系数中，仅出现在波函数的自变量中；在哈密顿约束算符作用下，它们仅被仅依赖于体积的因子缩放。得益于这一性质，Bianchi-I 希尔伯特空间中的物理态天然存在一个到各向同性希尔伯特空间中物理态的投影。积分掉各向异性自由度后，Bianchi-I 模型的量子差分方程可以精确约化为各向同性 $k = 0$ FLRW 模型的量子差分方程 [59]。这为“如何从更一般的量子动力学得到对称性约化后的量子动力学”提供了一个典型范例。考虑到 BKL 猜想断言广义相对论在类空奇点附近的动力学可以很好地用 Bianchi-I 宇宙学建模，Bianchi-I 模型到各向同性 FLRW 模型的投影表明，后者或许能保留完整理论中均匀各向同性分支量子动力学的一些基本特征。最后可以证明，当量子几何效应可忽略时，Wheeler-DeWitt 微分方程是 Bianchi-I 量子差分方程的良好近似。但在普朗克能区，该近似失效，且和各向同性情况一样，两种量子理论给出截然不同的物理预言。

Similar to the isotropic models, the effective description of the quantum dynamics is also shown to be numerically valid for the Bianchi-I model when the background quantum states are highly peaked around the classical trajectories at late times [14, 157]. A caveat is that this analysis has been so far performed only for the Madrid prescription in the vacuum case. These semiclassical states remain peaked throughout the whole evolution of the background spacetime, and hence the effective dynamics also faithfully captures the quantum geometric effects in the Planck regime. Therefore, the effective dynamics is widely used to explore the cosmological implications of the quantum dynamics of the Bianchi-I universe. The corresponding effective Hamiltonian constraint in the $\bar{\mu}$ scheme is given explicitly by [58, 59]

与各向同性模型类似，对于 Bianchi-I 模型，当背景量子态在后期高度集中在经典轨迹附近时，量子动力学的有效描述也被证明在数值上成立 [14, 157]。需要注意的是，到目前为止该分析仅针对真空情况的马德里处方得到。这些半经典态在背景时空的整个演化过程中始终保持集中，因此有效动力学也能可靠地捕捉普朗克能区的量子几何效应。因此，有效动力学被广泛用于研究 Bianchi-I 宇宙量子动力学的宇宙学意义。$\bar{\mu}$ 方案下对应的有效哈密顿约束已被明确给出，即 [58, 59]

$$\mathcal{H}^{\text{eff}} = -\frac{1}{8\pi G\gamma^2 v}\left(\frac{\sin\left(\bar{\mu}_1 c_1\right)}{\bar{\mu}_1}\frac{\sin\left(\bar{\mu}_2 c_2\right)}{\bar{\mu}_1} p_1 p_2 + \text{ cyclic terms }\right) + v\rho, \tag{52}$$

where $\bar{\mu}_1, \bar{\mu}_2$ and $\bar{\mu}_3$ are given in (50). In the isotropic limit when $p_1 = \ p_2 = p_3$, the above effective Hamiltonian constraint reduces to the one in the isotropic model. Using the Poisson bracket between connection and triad variables, $\{c_i, p_j\} = 8\pi G\gamma\delta_{ij}$, it is straightforward to derive the Hamilton’s equations with non-perturbative quantum gravity modifications. For c_1 and p_1 variables, these turn out to be

其中 $\bar{\mu}_1, \bar{\mu}_2$ 和 $\bar{\mu}_3$ 由式 (50) 给出。在各向同性极限下，当 $p_1 = \ p_2 = p_3$ 时，上述有效哈密顿约束退化为各向同性模型中的形式。利用联络与标架变量之间的泊松括号 $\{c_i, p_j\} = 8\pi G\gamma\delta_{ij}$ ，可以直接推导得到包含非微扰量子引力修正的哈密顿方程。对于变量 c_1 和 p_1 ，方程形式为

$$\dot{p_1} = \frac{p_1}{\gamma\lambda}\left(\sin\left(\bar{\mu}_2 c_2\right) + \sin\left(\bar{\mu}_3 c_3\right)\right)\cos\left(\bar{\mu}_1 c_1\right), \tag{53}$$

$$\begin{aligned} \dot{c_1} = \frac{v}{2\gamma\lambda^2 p_1} & \left[c_2\bar{\mu}_2\cos\left(\bar{\mu}_2 c_2\right)\left(\sin\left(\bar{\mu}_3 c_3\right) + \sin\left(\bar{\mu}_1 c_1\right)\right)\right. \\ & + c_3\bar{\mu}_3\cos\left(\bar{\mu}_3 c_3\right)\left(\sin\left(\bar{\mu}_1 c_1\right) + \sin\left(\bar{\mu}_2 c_2\right)\right) \\ & - c_1\bar{\mu}_1\cos\left(\bar{\mu}_1 c_1\right)\left(\sin\left(\bar{\mu}_2 c_2\right) + \sin\left(\bar{\mu}_3 c_3\right)\right) \\ & - \left(\sin\left(\bar{\mu}_1 c_1\right)\sin\left(\bar{\mu}_2 c_2\right) + \sin\left(\bar{\mu}_2 c_2\right)\sin\left(\bar{\mu}_3 c_3\right)\right. \\ & \left.\left. + \sin\left(\bar{\mu}_3 c_3\right)\sin\left(\bar{\mu}_1 c_1\right)\right)\right] + 8\pi G\gamma\frac{\partial v\rho}{\partial p_1}. \end{aligned} \tag{54}$$

Similarly one can obtain equations of motion for other connection and triad variables. In the general case when the anisotropies are present, the effective dynamics governed by the above Hamilton’s equations leads to a bounded energy density with its maximum coinciding with the maximum energy density in the isotropic $k = 0$ case. Due to mathematical complexity, a modified generalized Friedmann equation in terms of the energy density and the anisotropic shear is currently not available for the Bianchi-I model, but analyses based on the Hamilton's equations reveal the upper bounds on mean Hubble rate and the anisotropic shear [163] which turn out to be $H_{\max} = 1/(2\gamma\lambda)$ and $\sigma^2_{\max} = 10.125/\left(3\gamma^2\lambda^2\right)$. Interestingly, numerical simulations for different types of matter and potentials in the Bianchi-I model reveal a novel seemingly universal parabolic relationship between the energy density and shear at the bounce which can shed important insights on the generalized Friedmann equation with quantum geometric modifications [185]. Further, these investigations reveal that the anisotropic shear at the bounce reaches its maximum value when the energy density at the bounce reaches approximately half of its maximum allowed value.

同理可得其他联络和标架变量的运动方程。一般情况下, 当存在各向异性时, 上述哈密顿方程支配的有效动力学使得能量密度有界，其最大值与各向同性 $k = 0$ 情形下的最大能量密度一致。受限于数学复杂性，目前 Bianchi-I 模型尚未得到以能量密度和各向异性剪切表示的修正广义弗里德曼方程,但基于哈密顿方程的分析得到了平均哈勃率和各向异性剪切的上界 [163]，分别为 $H_{\max} = 1/(2\gamma\lambda)$ 和 $\sigma^2_{\max} = 10.125/\left(3\gamma^2\lambda^2\right)$。有趣的是，对 Bianchi-I 模型中不同类型物质和势的数值模拟发现，反弹点处能量密度与剪切之间存在一种新颖且看似普适的抛物线关系, 这可为研究包含量子几何修正的广义弗里德曼方程提供重要启示 [185]。此外，上述研究还发现，当反弹点的能量密度达到其最大允许值的约一半时，反弹处的各向异性剪切达到最大值。

The evolution of the loop quantized Bianchi-I universe turns out to be nonsingular for various types of matter with a bounce taking place at the minimal mean scale factor, see for example Fig. 3 for a Bianchi-I universe filled with a massive scalar field. In particular, it can be shown that all the curvature invariants are bounded, and the strong singularities are resolved throughout the non-singular evolution of the Bianchi-I universe as long as the null energy condition is satisfied [24]. The geometric structures of the bounces are also richer than those in the isotropic case where only the point-like bounce appears. In contrast, in the presence of anisotropies, the bounces can have barrel-, pancake- and cigar-like structures which can be obtained from a backward/forward evolution of the effective dynamics in the expanding/contracting phase using the same initial conditions in the classical regime that can lead to the singularities of the same type in GR. Therefore, these bounces can be regarded as the finite versions of the different types of the classical singularities which are resolved due to the underlying discreteness of the quantum geometry. Moreover, the quantum geometric effects also play the role of bridging different geometric structures in the pre- and post-bounce phases, leading to a Kasner transition that is impossible in the classical theory [190, 193, 194]. These Kasner transitions respect specific selection rules depending on the matter content of the universe. For example, in a Bianchi-I universe filled with the dust and radiation, the structure of the universe when approaching the bounce must be cigar-like on at least one side of the bounce. Therefore, in such a universe there is no pancake-pancake transition across the bounce. On the other hand, in a Bianchi-I universe filled with the stiff matter, there is no pancake-like structure on either side of the bounce, and depending on the anisotropic parameters, certain types of the transitions are more favored over the other types [190]. Finally, the inflationary and ekpyrotic scenarios have been studied in the setting of loop quantized Bianchi-I models. In particular, inflationary attractors for ϕ^2 inflation have been found in the Bianchi-I model in LQC, and it has been shown that there exists a quantum bounce in the past evolution of the inflationary trajectories and anisotropies remain bounded throughout the evolution [37]. Similarly, non-singular ekpyrotic model has been constructed using effective dynamics of Bianchi-I LQC [49].

结果表明，圈量子化 Bianchi-I 宇宙对各类物质都是非奇异演化，在平均尺度因子取最小值时发生反弹，例如参见图 3 中充满有质量标量场的 Bianchi-I 宇宙。特别地，可以证明只要满足零能量条件，所有曲率不变量都是有界的，Bianchi-I 宇宙在整个非奇异演化过程中强奇点都被消解 [24]。反弹的几何结构也比仅存在点反弹的各向同性情形更丰富。与之相反，存在各向异性时，反弹可呈现桶状、饼状和雪茄状结构: 在经典区域给定相同初始条件，广义相对论中该初始条件会导致对应类型的奇点，而在有效动力学的膨胀/收缩阶段做向后/向前演化即可得到这些不同结构的反弹。因此，这些反弹可看作各类经典奇点的有限形式，量子几何的本征离散性消解了这些经典奇点。此外，量子几何效应还起到连接反弹前后不同几何结构的作用，产生了经典理论中不可能存在的卡斯纳跃迁 [190, 193, 194]。卡斯纳跃迁遵循依赖于宇宙物质组分的特定选择规则。例如，在充满尘埃和辐射的 Bianchi-I 宇宙中，趋近反弹时宇宙至少在反弹的一侧必须呈雪茄状结构，因此这类宇宙中不会存在跨反弹的饼-饼跃迁。另一方面，在充满刚性物质的 Bianchi-I 宇宙中，反弹两侧都不存在饼状结构，且根据各向异性参数的不同，某些类型的跃迁相比其他类型更易发生 [190]。最后，研究者已经在圈量子化 Bianchi-I 模型框架下研究了暴胀和火劫场景。特别地，在圈量子宇宙学 (LQC) 的 Bianchi-I 模型中已经找到了 ϕ^2 暴胀的暴胀吸引子，结果表明暴胀轨迹的过去演化中存在量子反弹，且各向异性在整个演化过程中始终保持有界 [37]。类似地，研究者利用 Bianchi-I LQC 的有效动力学构造了非奇异火劫模型 [49]。

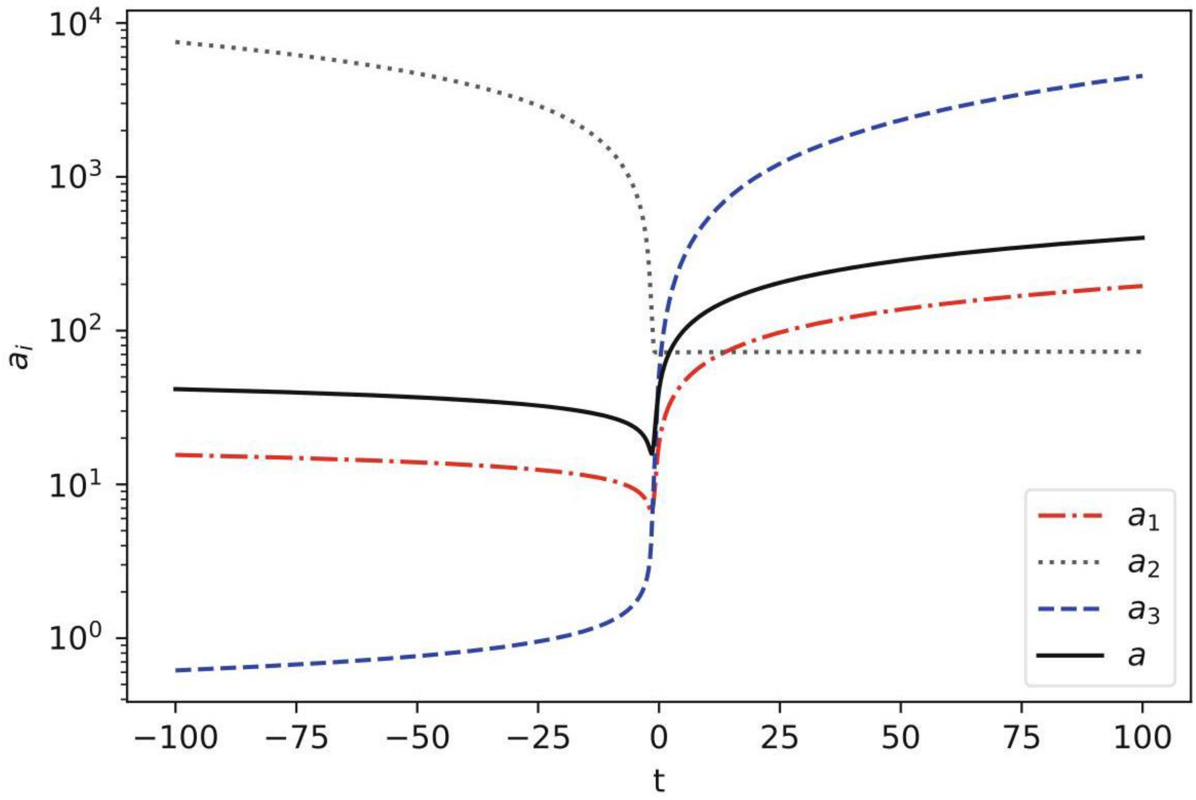

Fig. 3 Non-singular evolution of scale factors in the loop quantized Bianchi-I spacetime is shown for the case of an inflationary potential. The approach to classical singularity is cigar-like with two scale factors decreasing and one increasing. The resulting shape of the bounce is thus not point-like but as a cigar. Singularity is resolved with finite values of energy density and anisotropic shear

图 3 展示了暴涨势情况下，圈量子化 Bianchi-I 时空中标度因子的非奇异演化。趋近经典奇点时的形态为雪茄型: 两个标度因子减小，一个增大。因此反弹的最终形态并非点型，而是雪茄型。奇点得到解决，能量密度与各向异性剪切均为有限值。

Fock Quantized Inhomogeneities in Polymer Background: Polarized Gowdy Models

聚合物背景下的福克量子化非均匀性: 极化 Gowdy 模型

In addition to the cosmological spacetimes in the mini-superspace characterized by finite degrees of freedom, loop quantization has also been applied to those in the midi-superspace where inhomogeneities are also present as infinite degrees of freedom of the system [62-67]. The motivation to study these spacetimes is to check the robustness of the physical implications, such as the singularity resolution, resulting from the quantum geometry effects, as well as to investigate the impacts of the continuous degrees of freedom on the qualitative dynamics of LQC universe. One of the examples of this type is the Gowdy models which describe globally hyperbolic spacetimes with two space-like commuting Killing vector fields and compact spatial sections [195]. The simplest one of the Gowdy models is the linearly polarized Gowdy T^3 model. Since the Killing vector fields are also orthogonal to the hypersurface in this case, the spatial section T^3 can be described by coordinates (ζ, η, δ) with $\zeta, \eta, \delta \in S^1$, and thus two Killing vector fields are ∂_η and ∂_δ . Since the metric components which only depend on t and ζ are periodic in ζ , they can be expanded in the Fourier series in the ζ -momentum space. Using the symmetry of the system, one can completely fix the gauge freedom associated with the diffeomorphism constraints in η and δ [63], leaving alone two global constraints: the diffeomorphism constraint C_ζ which generates translation in ζ and the Hamiltonian constraint C_H which consists of a homogeneous part C_{hom} and an inhomogeneous part C_{inh} .

除了有限自由度描述的微超空间宇宙时空外，圈量子化也已应用于中微超空间的时空，在这类时空中非均匀性也作为系统的无限自由度存在 [62-67]。研究这类时空的动机，一是检验量子几何效应带来的奇点解决等物理结论的稳健性，二是探究连续自由度对圈量子宇宙学宇宙定性动力学的影响。这类模型的一个例子是 Gowdy 模型，它描述具有两个对易类空 Killing 向量场和紧致空间切片的整体双曲时空 [195]。最简单的 Gowdy 模型是线极化 Gowdy T^3 模型。由于在该情形下 Killing 向量场也与超曲面对正交，空间切片 T^3 可由坐标 (ζ, η, δ) 描述，满足 $\zeta, \eta, \delta \in S^1$ ，因此两个 Killing 向量场为 ∂_η 和 ∂_δ 。由于度量分量仅依赖于 t 和 ζ ，且在 ζ 上具有周期性，因此可以在 ζ 动量空间中展开为傅里叶级数。利用系统的对称性，可以完全固定与 η 和 δ 中微分同胚约束相关的规范自由度 [63]，仅剩下两个全局约束: 生成 ζ 平移的微分同胚约束 C_ζ ， 以及由齐次部分 C_{hom} 和非齐次部分 C_{inh} 构成的哈密顿约束 C_H 。

The loop quantization of the Gowdy T^3 model adopts a hybrid quantization method, where the homogeneous sector is loop quantized, while the inhomogeneous degrees of freedom are Fock quantized. A complete loop quantization requires a polymer quantization of the inhomogeneities which remains to be achieved. In the hybrid approach, the classical diffeomorphism and the Hamiltonian constraints are expressed in terms of the variables adapted to this quantization scheme. In particular, since the homogeneous sector of the Gowdy T^3 model is equivalent to the vacuum Bianchi-I spacetime, it is appropriate to be parameterized by the Ashtekar-Barbero connection c_i , with $i = \zeta, \eta, \delta$, and the corresponding densitized triads p_i introduced in section ”The Hamiltonian Formulation of Anisotropic Spacetime: Bianchi-I Model”. As for the inhomogeneous degrees of freedom, Fock quantization scheme prefers the use of the creation and annihilation variables (a_m, a_m^*) , where m can be any nonzero integer, standing for the nonzero modes that can be naturally associated with a free massless scalar field. In terms of the chosen variables, the expressions of the diffeomorphism and the Hamiltonian constraints can be obtained, which are given explicitly in [65]. Then using the techniques described in the above subsection for Bianchi-I spacetime and promoting the creation and annihilation variables to their operator analogs, the Hamiltonian constraint operator, a complete set of the Dirac observables as well as the physical inner product can be constructed precisely following the same strategy as used in the loop quantization of the spatially flat FLRW spacetime. The annihilation of the physical states by the Hamiltonian constraint operator also leads to a quantum difference equation with uniform steps in the volume. As a result, the superselection sector in the kinematic Hilbert space of the Bianchi-I spacetime is also retained in the Gowdy T^3 model. Moreover, the zero volume states in the homogeneous sector turn out to be decoupled by the action of the Hamiltonian constraint. As a result, the singularity is resolved in the quantum theory. Another interesting property of the quantized Gowdy T^3 model is that its physical Hilbert space is a tensor product of the physical Hilbert space of the Bianchi-I model and a Fock space in the standard Fock quantization, implying that the standard quantum field theory is actually recovered in the framework of loop quantization. This provides a prototype for considering the perturbations over a loop quantized cosmological background. Finally, the Planck physics in the Gowdy T^3 model has also been explored by using the effective dynamics of LQC, which again confirms that the singularity is really replaced by the quantum bounce and one of the effects of the inhomogeneities is to increase the bounce volume as compared with the LQC Bianchi-I spacetime [64].

Gowdy T^3 模型的圈量子化采用混合量子化方法，其中均匀部分采用圈量子化，而非均匀自由度采用福克量子化。完整的圈量子化要求对非均匀部分也采用聚合物量子化，这一点目前尚未实现。在混合方法中，经典微分同胚约束和哈密顿约束以适配该量子化方案的变量表示。具体而言，由于 Gowdy T^3 模型的均匀部分等价于真空 Bianchi-I 时空，适合用 Ashtekar-Barbero 联络 c_i 参数化，其中满足 $i = \zeta, \eta, \delta$ ，对应的密化标架 p_i 已在“各向异性时空的哈密顿表述:Bianchi-I 模型”一节引入。对于非均匀自由度，福克量子化方案倾向于使用产生和湮灭变量 (a_m, a_m^*) ，其中 m 可取任意非零整数，对应可自然关联到自由无质量标量场的非零模式。利用所选变量可得到微分同胚约束和哈密顿约束的表达式，具体可见文献 [65]。随后借助上文小节中处理 Bianchi-I 时空的技术，将产生和湮灭变量提升为对应算符，即可遵循空间平坦 FLRW 时空圈量子化的相同策略，严格构造哈密顿约束算符、全套狄拉克可观测量以及物理内积。哈密顿约束算符对物理态的湮灭作用同样会得到一个体积步长均匀的量子差分方程。因此，Bianchi-I 时空运动学希尔伯特空间中的超选择分支也在 Gowdy T^3 模型中保留了下来。此外，均匀部分的零体积态会被哈密顿约束的作用解耦，奇异性在量子理论中得以解决。量子化后的 Gowdy T^3 模型另一个有趣的性质是，其物理希尔伯特空间是 Bianchi-I 模型物理希尔伯特空间与标准福克量子化中福克空间的张量积，这意味着标准量子场论实际上在圈量子化框架下得到了恢复，为研究圈量子化宇宙学背景上的微扰提供了原型。最后，研究者还利用 LQC 的有效动力学探索了 Gowdy T^3 模型中的普朗克物理，再次证实奇异性确实被量子反弹取代，且与 LQC 框架下的 Bianchi-I 时空相比，非均匀性的其中一个效应是增大了反弹体积 [64]。

Beyond Standard LQC: Incorporating Additional Elements from LQG

超越标准圈量子宇宙学: 融入圈量子引力的额外元素

So far we have discussed various results in different cosmological models, including in the presence of anisotropies and inhomogeneities which show the robustness of singularity resolution in LQC. Let us here note that since LQC is not derived from LQG, it is important to understand the way physics of Planck scale changes when we include more elements from LQG. The goal of this section is to understand two such inputs and the way they affect predictions in LQC. We overview two variant cosmological models for the spatially flat FLRW universe that originate from a separate treatment of the Lorentzian term in the classical Hamiltonian constraint of GR. These models can be regarded as extensions of standard LQC due to different quantization prescriptions. As discussed in section "Loop Quantization of the Spatially Flat FLRW Universe", in the spatially flat LQC, due to underlying symmetry the Euclidean and Lorentzian terms are combined before quantization. But how do the quantum constraint and its predictions change when we treat them separately? An independent treatment of the Lorentzian term using the Thiemann regularization was first discussed in [74] where the corresponding quantum Hamiltonian constraints resulting from two different ways of regularization of the Lorentzian term were derived. In the following, these two variants are called mLQC-I and mLQC-II, meaning the modified LQC model I and II, following the convention in [77]. Later, mLQC-I was rediscovered in [75] from computing the expectation value of the quantum Hamiltonian constraint in LQG by using the complexifier coherent states developed by Thiemann and Winkler [196, 197]. Different from standard LQC, the quantum difference equations in mLQC-I/II turn out to be 4th-order difference equations [76] which lead to more complicated structure of the effective Hamiltonian constraints. This also results in a generic resolution of strong curvature singularities [198]. The correct forms of the modified Friedmann and Raychaudhuri equations in mLQC-I were found in [77] implying an emergent quasi de-Sitter phase in the contracting branch [199, 200] with a rescaled Newton's constant [77]. On the other hand, the dynamics of mLQC-II turns out to be very similar to standard LQC with a symmetric bounce and modified maximum energy density. The phenomenological implications of these two models are extensively discussed in the literature, including both the background dynamics and the linear perturbations [173, 174, 201 − 205] . In the following, we start with the effective dynamics of mLQC-I/II and then briefly mention their implications on the inflationary scenario.

至此我们已经讨论了不同宇宙学模型中的诸多结果，其中包括各向异性和非均匀性存在的情况，这些结果都体现了圈量子宇宙学中奇点消解的稳健性。在此我们需要指出，由于圈量子宇宙学并非直接从圈量子引力导出，因此理解当我们引入更多来自圈量子引力的元素时普朗克尺度物理会如何变化十分重要。本节的目标是探究两个这类引入的元素，以及它们如何影响圈量子宇宙学的预言。我们概述两类空间平坦 FLRW 宇宙的变体宇宙学模型，这类模型源于对广义相对论经典哈密顿约束中洛伦兹项的分开处理。由于采用了不同的量子化方案，这些模型可被视为标准圈量子宇宙学的推广。正如“空间平坦 FLRW 宇宙的圈量子化”一节所述，在空间平坦圈量子宇宙学中，由于底层对称性，欧几里得项与洛伦兹项在量子化前已经被合并。但如果我们分开处理这两项，量子约束及其预言会发生怎样的变化？文献 [74] 最早讨论了使用蒂姆正则化独立处理洛伦兹项，并推导了洛伦兹项两种不同正则化方式得到的对应量子哈密顿约束。遵循文献 [77] 的约定，我们将这两种变体称为 mLQC-I 和 mLQC-II，即修正圈量子宇宙学模型 I 和模型 II。后来，文献 [75] 利用蒂姆和温克勒提出的复相干态 [196, 197] 计算圈量子引力中量子哈密顿约束的期望值，重新得到了 mLQC-I。与标准圈量子宇宙学不同，mLQC-I/II 中的量子差分方程为四阶差分方程 [76]，这使得有效哈密顿约束的结构更为复杂，也带来了强曲率奇点的一般性消解 [198]。文献 [77] 得到了 mLQC-I 中修正弗里德曼方程和修正雷乔杜里方程的正确形式，结果表明收缩分支中会出现涌现类德西特相 [199, 200]，同时牛顿常数被重整 [77]。另一方面，mLQC-II 的动力学被证明与标准圈量子宇宙学非常相似，同样存在对称反弹，仅最大能量密度得到修正。这两类模型的唯象影响已有大量文献讨论，涵盖背景动力学和线性扰动 [173, 174, 201 − 205] 。下文我们将首先介绍 mLQC-I/II 的有效动力学，随后简要说明它们对暴涨图景的影响。

The Effective Dynamics of mLQC-I/II

mLQC-I/II 的有效动力学

Similar to the derivation of the effective dynamics in LQC, using the sharply peaked Gaussian coherent states, one can obtain the effective Hamiltonian constraint of mLQC-I/II by computing the expectation value of the Hamiltonian constraint operator in each model. In mLQC-I, the effective Hamiltonian constraint takes the form [74,75]

与推导圈量子宇宙学有效动力学的过程类似，利用尖峰高斯相干态，通过计算每个模型中哈密顿约束算符的期望值，可以得到 mLQC-I/II 的有效哈密顿约束。在 mLQC-I 中，有效哈密顿约束的形式如下 [74,75]

$$\mathcal{H}^{\mathrm{I}} = \frac{3v}{8\pi G\lambda^2}\left\{\sin^2(\lambda b) - \frac{(\gamma^2+1)\sin^2(2\lambda b)}{4\gamma^2}\right\} + \mathcal{H}_M. \tag{55}$$

With the help of the Poisson bracket $\{b, v\} = 4\pi G\gamma$, it is straightforward to derive the Hamilton's equations of the system. The novel property of mLQC-I shows up when we try to derive the modified Friedmann equation from the Hamilton's equation of the volume. From the Hamiltonian constraint, one can find two branches, i.e., the $b_\pm$ branches with the relations

借助泊松括号 $\{b, v\} = 4\pi G\gamma$ ，可以直接推导出系统的哈密顿方程。当我们从体积的哈密顿方程推导修正弗里德曼方程时，就能体现出 mLQC-I 的全新性质。从哈密顿约束可以得到两个分支，即 $b_\pm$ 分支，满足关系

$$\sin^2(\lambda b_\pm) = \frac{1 \pm \sqrt{1 - \rho/\rho_c^{\mathrm{I}}}}{2(\gamma^2 + 1)}, \tag{56}$$

where $\rho_c^{\mathrm{I}} \equiv 3/\left[32\pi G\lambda^2\gamma^2(\gamma^2+1)\right]$ is the maximum energy density that can be attained in the model. The difference between these two branches can be seen directly from the above relation. In particular, when the energy density tends to vanish, b_- goes to zero, while b_+ approaches a nonzero value. This implies that the classical limit can only be recovered in the b_- branch. A detailed analysis on the modified Friedmann equation confirms this intuition. Substituting the relation (56) into the Hamilton's equation of the volume and then squaring it, one can obtain the modified Friedmann equation for each branch [77]. For the b_- branch, the modified Friedmann equation turns out to be

其中 $\rho_c^{\mathrm{I}} \equiv 3/\left[32\pi G\lambda^2\gamma^2(\gamma^2+1)\right]$ 是该模型中可达到的最大能量密度。两个分支的差异可以从上述关系直接看出。具体来说，当能量密度趋于零时，b_- 趋于零，而 b_+ 趋近于一个非零值。这说明只有 b_- 分支可以恢复经典极限。对修正弗里德曼方程的详细分析证实了这一推测。将关系式 (56) 代入体积的哈密顿方程后平方，即可得到每个分支的修正弗里德曼方程 [77]。对于 b_- 分支，修正弗里德曼方程为

$$H_-^2 = \frac{8\pi G\rho}{3}\left(1 - \frac{\rho}{\rho_c^{\mathrm{I}}}\right)\left[1 + \frac{\gamma^2}{\gamma^2+1}\left(\frac{\sqrt{\rho/\rho_c^{\mathrm{I}}}}{1 + \sqrt{1 - \rho/\rho_c^{\mathrm{I}}}}\right)^2\right], \tag{57}$$

which asymptotes to the classical Friedmann equation as $\rho \ll \rho_c^{\mathrm{I}}$. The b_- branch describes the expanding phase of the universe in mLQC-I. In the backward evolution of the universe, the volume decreases continuously with an increasing energy density. When ρ increases to ρ_c^{I}, a quantum bounce takes place irrespective of the matter content. In this process, similar to the standard LQC, when the energy density increases to a threshold value in the Planck regime (less than ρ_c^{I}), the universe enters into a super-inflationary phase in which $\dot{H} > 0$. In mLQC-I, this threshold value of the energy density depends on the Barbero-Immirzi parameter γ, and its numerical value is about $\rho_s^- \approx 0.503\rho_c^{\mathrm{I}}$, with $\gamma \approx 0.2375$. On the other hand, the contracting phase of the universe in mLQC-I is described by the b_+ branch in which the modified Friedmann equation is given by

当 $\rho \ll \rho_c^{\mathrm{I}}$ 时，它渐近趋近于经典弗里德曼方程。b_- 分支描述了 mLQC-I 中宇宙的膨胀阶段。在宇宙的逆向演化中，体积不断缩小，能量密度不断升高。当 ρ 增大到 ρ_c^{I} 时，无论物质组分如何，都会发生量子反弹。这一过程中，和标准圈量子宇宙学类似，当能量密度升高到普朗克能标的一个阈值 (小于 ρ_c^{I}) 时，宇宙进入超膨胀阶段，此时 $\dot{H} > 0$ 。在 mLQC-I 中，这个能量密度阈值依赖于巴贝罗-伊姆米尔齐参数 γ ，其数值约为 $\rho_s^- \approx 0.503\rho_c^{\mathrm{I}}$ ，其中 $\gamma \approx 0.2375$ 。另一方面，mLQC-I 中宇宙的收缩阶段由 b_+ 分支描述，该分支的修正弗里德曼方程为

$$H_+^2 = \frac{8\pi G_\alpha\rho_\Lambda}{3}\left(1 - \frac{\rho}{\rho_c^{\mathrm{I}}}\right)\left[1 + \left(\frac{1 - 2\gamma^2 + \sqrt{1 - \rho/\rho_c^{\mathrm{I}}}}{4\gamma^2\left(1 + \sqrt{1 - \rho/\rho_c^{\mathrm{I}}}\right)}\right)\frac{\rho}{\rho_c^{\mathrm{I}}}\right], \tag{58}$$

where the rescaled Newton’s constant $G_\alpha \equiv G\left(1-5\gamma^2\right)/\left(\gamma^2+1\right) \approx 0.680G$ and the emergent cosmological constant $\rho_\Lambda \equiv 3/\left[8\pi G\alpha\lambda^2\left(1+\gamma^2\right)^2\right] \approx 0.0304$. Hence, this branch is characterized by an emergent quasi-de Sitter phase with a rescaled Newton’s constant at zero energy density. When $\rho \to 0$, the Hubble rate $H^2 \to \frac{8\pi G\alpha\rho_\Lambda}{3} \approx 0.173$, which is almost of Planck scale. Thus, the evolution of the universe in mLQC-I is asymmetric with respect to the quantum bounce (If instead of working with cosmic time, one chooses massless scalar field as a clock, one reaches infinite volume at a finite value of this clock. With such a clock, quantum extensions in the physical Hilbert space have been explored assuming a massless scalar field as the only matter content in Ref. [200].). Surprisingly, in the b_+ branch, the Hubble rate becomes vanishing at the minimum energy density $\rho_{\min} = -3/\left(8\pi G\lambda^2\right) \approx -0.023$, which is a negative number. As a result, a necessary condition for realization of a cyclic universe in mLQC-I is the violation of the weak energy condition [188]. The super-inflationary phase in the b_+ branch begins when the energy density reaches a different threshold value $\rho_s^+ \approx 0.377\rho_c^{\mathrm{I}}$. Therefore, the super-inflationary regime in mLQC-I is also asymmetric with respect to the quantum bounce. Considering all the remarkable properties of its contracting branch, mLQC-I serves as one of the concrete examples where different quantization prescriptions can result in qualitatively distinct quantum dynamics. In comparison to the novel properties of mLQC-I, the qualitative dynamics of mLQC-II resembles that of the standard LQC. In particular, the evolution of the universe filled with a massless field is also symmetric with respect to the bounce. The effective dynamics of mLQC-II is governed by the modified Friedmann equation [173]

其中包含重标度牛顿常数 $G_\alpha \equiv G\left(1-5\gamma^2\right)/\left(\gamma^2+1\right) \approx 0.680G$ 和涌现宇宙学常数 $\rho_\Lambda \equiv 3/\left[8\pi G\alpha\lambda^2\left(1+\gamma^2\right)^2\right] \approx 0.0304$ 。因此，该分支的特征是零能量密度下存在具有重标度牛顿常数的涌现类德西特相。当 $\rho \to 0$ 时，哈勃率为 $H^2 \to \frac{8\pi G\alpha\rho_\Lambda}{3} \approx 0.173$ ，几乎达到普朗克尺度。因此，mLQC-I 中宇宙的演化相对于量子反弹是不对称的 (如果不使用宇宙时间，而是选择无质量标量场作为时钟，会在该时钟取有限值时达到无限体积。在该时钟设定下，文献 [200] 假设无质量标量场是唯一物质组分，已经研究了物理希尔伯特空间的量子延拓)。令人惊讶的是，在 b_+ 分支中，哈勃率在最小能量密度 $\rho_{\min} = -3/\left(8\pi G\lambda^2\right) \approx -0.023$ 处趋近于零，而 $\rho_{\min} = -3/\left(8\pi G\lambda^2\right) \approx -0.023$ 是一个负数。因此，mLQC-I 中实现循环宇宙的必要条件是弱能量条件破缺 [188]。当能量密度达到另一阈值 $\rho_s^+ \approx 0.377\rho_c^{\mathrm{I}}$ 时，b_+ 分支的超暴胀阶段开始。因此，mLQC-I 的超暴胀区域相对于量子反弹也是不对称的。考虑到其收缩分支的所有显著性质，mLQC-I 是不同量子化方案可产生定性不同量子动力学的具体例子之一。与 mLQC-I 的新奇性质相比，mLQC-II 的定性动力学与标准 LQC 相似。尤其是，填充无质量场的宇宙演化相对于反弹也是对称的。mLQC-II 的有效动力学由修正弗里德曼方程支配 [173]

$$H^2 = \frac{16\pi G\rho}{3}\left(1-\frac{\rho}{\rho_c^{\mathrm{II}}}\right)\left(\frac{1+4\gamma^2\left(\gamma^2+1\right)\rho/\rho_c^{\mathrm{II}}}{1+2\gamma^2\rho/\rho_c^{\mathrm{II}}+\sqrt{1+4\gamma^2\left(1+\gamma^2\right)\rho/\rho_c^{\mathrm{II}}}}\right), \tag{59}$$

with the maximum energy density $\rho_c^{\mathrm{II}} = 3\left(\gamma^2+1\right)/2\pi G\gamma^2\lambda^2$, for both contracting and expanding branches. The quantum bounce takes place generically at the maximum energy density $\rho = \rho_c^{\mathrm{II}}$, and the super-inflation occurs for $\rho \geq 0.513\rho_c^{\mathrm{II}}$. Although there is no qualitative distinctions between mLQC-II and standard LQC as discovered so far, it still implies that quantization ambiguities can result in a different maximum energy density and durations of the super-inflationary phase in the Planck regime, that is, except for some generic features such as the resolution of the big-bang singularity by a quantum bounce, the details of the quantum dynamics are also closely related with the specific quantization prescriptions.

对于收缩分支和膨胀分支，最大能量密度均为 $\rho_c^{\mathrm{II}} = 3\left(\gamma^2+1\right)/2\pi G\gamma^2\lambda^2$ 。量子反弹普遍发生在最大能量密度 $\rho = \rho_c^{\mathrm{II}}$ 处，且当 $\rho \geq 0.513\rho_c^{\mathrm{II}}$ 时发生超暴胀。尽管目前已发现 mLQC-II 和标准 LQC 之间不存在定性差异，但它仍表明量子化不确定性会导致普朗克区域中超暴胀阶段的最大能量密度和持续时间发生改变，也就是说，除了量子反弹解决大爆炸奇点这一共性特征外，量子动力学的细节也与具体量子化方案密切相关。

As the second part of this section, let us briefly summarize the results of the inflationary background dynamics in mLQC-I/II. Similar to standard LQC, the embedding of the inflationary paradigm into mLQC-I/II has been extensively studied in the literature [173, 174]. Taking into account an inflationary potential, it has been found that the inflationary phase is also a local attractor of the qualitative dynamics in each model. Detailed analysis has been done for the ϕ^2 potential, the fractional monodromy potential, the Starobinsky potential, the non-minimal Higgs potential, and the exponential potential. In particular, the numerical and analytical results of the background evolution with the chaotic and the Starobinsky potentials are available in mLQC-I/II. It has been shown explicitly that similar to standard LQC, for the kinetic-energy-dominated bounce, the evolution of the universe by the end of the inflationary (slow-roll) phase in the expanding branch can be divided into three distinctive stages: the bouncing phase, the transition phase, and the slow-roll phase. Following the approach used in LQC, the probability of the occurrence of the desired slow-roll phase is also estimated in mLQC-I/II: One can find that the probability for the desired slow-roll not to happen for a ϕ^2 potential is P^{I} (not realized) $\lesssim 1.12 \times 10^{-5}$ in mLQC-I and P^{II} (not realized) $\lesssim 2.62 \times 10^{-6}$ in mLQC-II.

作为本节的第二部分，我们简要总结 mLQC-I/II 中的暴胀背景动力学结果。与标准 LQC 类似，目前文献已对暴胀范式嵌入 mLQC-I/II 开展了大量研究 [173, 174]。考虑暴胀势后研究发现，在每个模型中，暴胀阶段同样是定性动力学的局部吸引子。目前已对 ϕ^2 势、分数单环绕势、斯塔罗宾斯基势、非最小希格斯势以及指数势开展了详细分析。特别地，mLQC-I/II 中已给出混沌势和斯塔罗宾斯基势下背景演化的数值与解析结果。研究明确表明，与标准 LQC 类似，对于动能主导的反弹，膨胀分支中暴胀 (慢滚) 阶段结束前的宇宙演化可分为三个截然不同的阶段: 反弹阶段、过渡阶段和慢滚阶段。沿用 LQC 中使用的方法，mLQC-I/II 也对理想慢滚阶段发生的概率进行了估计: 可以发现，对于 ϕ^2 势，不发生理想慢滚的概率在 mLQC-I 中为 P^{I} (未实现) $\lesssim 1.12 \times 10^{-5}$ ，在 mLQC-II 中为 P^{II} (未实现) $\lesssim 2.62 \times 10^{-6}$ 。

Though inflationary paradigm is compatible with both mLQC-I and mLQC-II, the same cannot be said about the alternatives of inflation based on cyclic models. A surprising result is that ekpyrotic models are incompatible with mLQC-I [188]. Note that this is not because mLQC-I does not result in singularity resolution rather the problem that comes from the lack of recollapse in the classical regime after one cycle and as a result multiple cycles cannot occur. Given the conventional wisdom that cyclic models are easy to construct with a non-singular bounce if one adds matter which causes a recollapse at late times, this result is indeed counter-intuitive but can be easily understood. As we have discussed, while standard LQC results in a classical regime in both pre-bounce and post-bounce epochs at large volumes, the same cannot be said for mLQC-I where the pre-bounce regime has a large effective cosmological constant phase originating from quantum geometry. It turns out that every bounce in mLQC-I switches the branch from b_+ to b_- (or vice versa). In the case of a model where there is a recollapse, it turns out that the asymptotic past and future branches, after the classical turnaround and the second bounce, are dictated by the b_+ branch. Therefore, unlike a model with single bounce in mLQC-I which connects b_+ and b_- branch, the additional classical turnaround and the subsequent bounce convert the future asymptotic branch after the second bounce into a b_+ branch. As discussed earlier the b_+ branch does not correspond to the classical spacetime with low curvature, which forbids another recollapse and a subsequent bounce. Hence, even in the presence of a non-singular bounce, a cyclic evolution with more than one cycle is not possible in mLQC-I unless one chooses a very large negative potential which overcomes Planckian effective cosmological constant. On the other hand, given the symmetric nature of bounce, cyclic models are as much compatible with mLQC-II as in standard LQC. An important question is whether this result implies that cyclic models are inconsistent with LQG. While this remains an open question, it is indeed true that highly asymmetric bounces leading to only one side of evolution with low spacetime curvatures are not just confined to mLQC-I but also occur in certain loop quantizations of Kantowski-Sachs spacetime [206]. One may conjecture that if LQG results in an evolution characterized by such an asymmetric bounce it would result in incompatibility with cyclic models.

尽管暴胀范式与 mLQC-I 和 mLQC-II 都兼容，但基于循环模型的暴胀替代方案却并非如此。一个令人惊讶的结论是: 火劫模型与 mLQC-I 不相容 [188]。注意这并非因为 mLQC-I 无法实现奇点 Resolution，问题反而出在: 一个循环结束后的经典区域不会发生坍缩，因此多个循环无法存在。传统观点认为，如果添加能在晚期引发坍缩的物质，很容易构造出具有非奇异反弹的循环模型，因此这一结果确实有违直觉，但不难理解: 正如我们已经讨论的，标准 LQC 中，大体积下反弹前后两个时期都能得到经典区域，但 mLQC-I 并非如此——它的反弹前区域存在一个源自量子几何的大有效宇宙学常数相。研究发现，mLQC-I 中每一次反弹都会将分支从 b_+ 切换为 b_- (反之亦然)。对于存在坍缩的模型，研究表明，在经典 turnaround 和第二次反弹后，渐进过去与未来分支都由 b_+ 分支决定。因此，与 mLQC-I 中连接 b_+ 和 b_- 分支的单次反弹模型不同，额外的经典 turnaround 及后续反弹会将第二次反弹后的未来渐进分支转换为 b_+ 分支。正如之前讨论的，b_+ 分支不对应低曲率的经典时空，这就阻止了再次坍缩和后续反弹的发生。因此，即使存在非奇异反弹，mLQC-I 中也不可能存在超过一个循环的循环演化，除非选择一个足够大的负势来抵消普朗克量级的有效宇宙学常数。另一方面，由于反弹具有对称性，循环模型和 mLQC-II 的兼容度与标准 LQC 一致。一个关键问题是: 这一结果是否意味着循环模型与 LQG 不自洽？尽管这仍然是一个开放问题，但一个不争的事实是: 能导致只有一侧存在低时空曲率演化的高度不对称反弹，并非仅存在于 mLQC-I，它也会出现在坎托夫斯基-萨克斯时空的某些圈量子化中 [206]。我们可以猜想: 如果 LQG 的演化以这种不对称反弹为特征，那么它就会与循环模型不相容。

Summary and Outlook

总结与展望

The development of LQC in the last two decades has made itself a prototype to exemplify how the underlying discrete quantum geometry effects can resolve strong curvature singularities intrinsic in the classical theory in various cosmological settings and meanwhile provide an insightful picture of the physics beyond the standard cosmology, especially of the quantum evolution of the universe in the Planck regime. Featured by the background independent formulation of the quantum theory, LQC makes use of the Ashtekar-Barbero variables tailored to symmetry reduced spacetimes to construct the quantum Hamiltonian constraint operator as well as relevant physical observables. Its kinematic Hilbert space is made up of essentially discrete quantum states which provide a quantum representation that is unitarily inequivalent to the Schrödinger representation. Another key element in the formulation is to introduce a minimal nonzero area gap to build up the nonlocal curvature operator. This becomes a vital step to yield a physically viable loop quantization which on one hand results in a non-singular evolution of the universe in the Planck regime and on the other hand asymptotes to the right classical limit in the low curvature regime. Its unique quantum representation makes itself distinct as compared with the Wheeler-Dewitt theory in the sense that LQC can only be well approximated by the Wheeler-Dewitt theory far away from the Planck regime.

> 过去二十年间圈量子宇宙学 (LQC) 的发展，已成为一个典范原型，说明了基础离散量子几何效应如何在多种宇宙学场景下解决经典理论固有的强曲率奇点，同时为标准宇宙学之外的物理，尤其是普朗克 regime 下的宇宙量子演化提供了富有洞察力的图景。LQC 以量子理论的背景无关表述为核心特征，借助适配对称性约化时空的阿西特卡-巴贝罗变量，构造了量子哈密顿约束算符以及相关的物理可观测量。其运动学希尔伯特空间本质上由离散量子态构成，这些量子态给出了与薛定谔表象幺正不等价的量子表象。表述中的另一关键要素是引入了一个最小非零面积间隙，以此构建非局域曲率算符。这一步是得到物理自洽的圈量子化的关键: 一方面，它使得普朗克 regime 下的宇宙实现非奇异演化；另一方面，在低曲率 regime，该理论渐近趋近于正确的经典极限。LQC 独特的量子表象使其有别于惠勒-德维特理论: 只有在远离普朗克 regime 的区域，LQC 才能被惠勒-德维特理论良好近似。

The resolution of big-bang singularity in LQC was first discovered in the simplest case of a homogeneous and isotropic FLRW universe filled with a massless scalar field where the numerical simulations of the quantum difference equation revealed a quantum bounce in the Planck regime [10]. Later, the robustness of the singularity resolution as well as the existence of the quantum bounce has been tested in more complicated contexts, such as spacetimes with a cosmological constant, nonvanishing spatial curvature, anisotropies, continuous degrees of freedom, and even in some modified gravity theories [207, 208]. Supported by the numerical simulations of the quantum difference equation, the effective description of the quantum evolution of the LQC universe plays an important role in achieving an analytical understanding of the singularity resolution in the Planck regime in various types of loop quantized spacetimes [23, 25]. Moreover, using the effective dynamics of LQC, well-known scenarios in the standard cosmology, such as the inflationary paradigm, the ekpyrotic- and matter-bounce scenarios, have been extensively studied with a purpose of providing a ultraviolet complete description of these scenarios in the context of the LQC universe. In addition to these topics which we discussed earlier, quantum geometric effects in LQC have also been applied in order to obtain nonsingular evolution in string motivated scenarios such as pre-big-bang cosmology [209] and multiverses [210, 211]. The impact of the quantum geometry effects on the dynamical evolution of the universe in all these scenarios as well as their detectable characteristic signals become the central subjects to investigate at the phenomenological level. Among all of these progresses, it is worthwhile to point out that the inflationary phase is found to be a local attractor in the LQC universe with/without anisotropies when the initial conditions of the dynamical trajectories are set in the Planck regime and the probability for the occurrence of inflation can also be computed explicitly due to the availability of a distinguished physical measure of the parameter space right at the quantum bounce in a homogeneous and isotropic LQC universe.

LQC 对大爆炸奇点的解决最初是在最简单的场景中发现的: 即充满无质量标量场的均匀各向同性 FLRW 宇宙，对量子差分方程的数值模拟显示，普朗克 regime 下存在量子反弹 [10]。随后，奇点解决的鲁棒性以及量子反弹的存在性在更复杂的场景中得到了检验，例如带宇宙常数的时空、非零空间曲率时空、各向异性时空、连续自由度时空，甚至在部分修正引力理论中也完成了检验 [207, 208]。得到量子差分方程数值模拟支持的 LQC 宇宙量子演化有效描述，在分析理解各类圈量子化时空普朗克 regime 的奇点解决中发挥着重要作用 [23, 25]。此外，借助 LQC 的有效动力学，学界已对标准宇宙学中的经典场景，例如暴胀范式、火劫与物质反弹场景开展了广泛研究，目的是在 LQC 宇宙的框架下为这些场景提供一个紫外完备的描述。除了我们之前讨论过的这些主题，LQC 的量子几何效应也已被应用于获得弦论启发场景下的非奇异演化，例如前大爆炸宇宙学 [209] 和多重宇宙 [210, 211]。量子几何效应对所有这些场景中宇宙动力学演化的影响，以及这些效应可被探测的特征信号，已经成为唯象层面研究的核心课题。在所有这些进展中，值得指出的是: 当动力学轨迹的初始条件设定在普朗克 regime 时，暴胀相被证明是带/不带各向异性的 LQC 宇宙的局部吸引子；并且由于均匀各向同性 LQC 宇宙中量子反弹处参数空间存在一个明确的物理测度，我们可以显式计算暴胀发生的概率。

In addition to coupling to the massless scalar field, the quantum theory for a homogeneous and isotropic LQC universe filled with a massive scalar field can also be constructed with a similar procedure [96]. The novel feature in the presence of the potential is that the scalar field can no longer serve as a global clock. As a result, some other reference fields are required to play the role of the clocks. One of the strategies is that the quantization can be implemented in the reduced phase space formulated in the relational formalism in which the Dirac observables are explicitly constructed before quantization is carried out. To deparameterize the theory, one needs to introduce some matter fields, such as the dust fields or the Klein-Gordon scalar fields, which serve as the reference fields with respect to which physical observables are constructed explicitly. For the Gaussian and the Brown-Kuchař dust fields, the resulting physical Hamiltonian in the reduced phase space takes the same form as its analog in the classical phase space with classical phase space variables replaced by their observable counterparts in the reduced phase space. Moreover, the physical Hamiltonian is no longer constrained to vanish. One can then construct the quantum theory using the techniques well developed in LQC, and it turns out that the impacts of the different reference fields on the quantum dynamics of the LQC universe can be tuned as small as possible with a careful choice of the initial energy density of the reference fields. This line of research is aimed at solving the problem of time in a general setting of LQC beyond the limit of the massless scalar field, and it helps justify the effective dynamics of LQC with a massive scalar field which is used to prevail in the literature for a phenomenological investigation of the cosmological implications of the LQC universe. Apart from the treatment of the massive scalar field, we also believe that a consistent loop quantization of both the geometric and matter degrees of freedom on the same footing is also important in achieving a general picture of the quantum dynamics of the LQC universe in the Planck regime.

除了耦合无质量标量场之外，填充有质量标量场的均匀各向同性圈量子宇宙学 (LQC) 宇宙的量子理论也可以通过类似步骤构建 [96]。存在势能的新特点是，标量场无法再充当全局时钟，因此需要其他参考场来承担时钟的作用。其中一种策略是在关联形式体系构造的约化相空间中完成量子化，在量子化进行之前先显式构造狄拉克可观测量。为了对理论去参数化，需要引入尘埃场或克莱因-戈登场这类物质场作为参考场，基于参考场显式构造物理可观测量。对于高斯尘埃场和 Brown-Kuchař 尘埃场，约化相空间中最终得到的物理哈密顿量，其形式与经典相空间中的对应形式相同，仅经典相空间变量被替换为约化相空间中的可观测量对应物。此外，物理哈密顿量不再被约束为零。之后就可以利用 LQC 中已发展成熟的技术构建量子理论，研究表明，只要仔细选择参考场的初始能量密度，不同参考场对 LQC 宇宙量子动力学的影响可以调节到极小。该研究方向旨在解决 LQC 中超出无质量标量场极限的一般设定下的时间问题，也有助于佐证文献中普遍用于探究 LQC 宇宙宇宙学效应的有质量标量场 LQC 有效动力学的合理性。除了处理有质量标量场，我们还认为，对几何自由度和物质自由度在同等基础上进行自洽的圈量子化，对于得到普朗克能标下 LQC 宇宙量子动力学的整体图景也十分重要。

Meanwhile, recent progress has been made in the direction of exploring the alternative quantization prescriptions other than the one used in standard LQC. The motivation mainly originates from endeavors to unravel the relationship between LQC and LQG. Due to the non-commutativity between quantization and symmetry reduction, it is a long-standing question to ask that how many features LQC has really inherited from the cosmological sector of full LQG. The bottom-to-top approach to addressing this concern has led to alternative LQC models, such as mLQC-I/II. Although some generic features, such as singularity resolution and the existence of a quantum bounce which connects a contracting phase with an expanding one, still remain true in these modified LQC models, they also exhibit novel features of the quantum dynamics even in the simplest setting of a homogeneous and isotropic FLRW universe and provide some evidence to show that the quantum dynamics in the Planck regime can have much richer structures than originally proposed in standard LQC. In particular, the universe can evolve asymmetrically with respect to the quantum bounce, and the evolution of the contracting phase can also proceed in a de Sitter phase with a Planck-scale cosmological constant. The emergent de Sitter phase in the contracting phase has also been observed in other alternative quantization of the Hamiltonian constraint constructed from the Chern-Simons action [212, 213] where the big-bang singularity is once again resolved in the case of the spatially flat FLRW universe. Thus, whether the classical universe can be recovered on the other side of the bounce is now in question. Although a bouncing evolution can be shown as obtained from a specific model of loop quantum gravity [214], the above results imply that standard LQC may not capture all of the features of the quantum cosmological sector of LQG. In addition to the modified LQC models, other strategies to investigate the connection between LQC and LQG include the quantum-reduced loop quantum gravity [70], the group theory cosmology [71, 72], and the path integral approach [73]. It is remarkable to note that although the improved dynamics with the $\bar{\mu}$ scheme was initially found in the homogeneous and isotropic FLRW LQC universe, its validity has extended to the other spacetimes and quantization prescriptions. Moreover, the $\bar{\mu}$ scheme is also found in other approaches such as the quantum-reduced loop quantum gravity and the path integral approach. Therefore, the $\bar{\mu}$ scheme, which is the only viable prescription known so far in homogeneous and isotropic cosmological models of LQC, seems to be consistent even in a more general setting.

与此同时，探索标准 LQC 所用量子化方案之外的其他替代量子化方案方向近年来已取得进展。其研究动机主要来自厘清 LQC 与全圈量子引力 (LQG) 之间关系的研究。由于量子化与对称性约化之间不对易，LQC 究竟从全 LQG 的宇宙学部分继承了多少特性一直是一个悬而未决的问题。解决该问题的自下而上方法已经催生了替代 LQC 模型，例如 mLQC-I/II。尽管奇点解决、存在连接收缩相和膨胀相的量子反弹这些通用特性在这些修正 LQC 模型中仍然成立，但即使在最简单的均匀各向同性 FLRW 宇宙设定下，它们也展现出了量子动力学的新特性，并提供证据表明普朗克能标的量子动力学可以拥有比标准 LQC 最初提出的更丰富的结构。尤其是，宇宙相对于量子反弹可以不对称演化，收缩相的演化也可以在具有普朗克尺度宇宙学常数的德西特相中进行。收缩相中涌现的德西特相也在由陈-西蒙斯作用量构造的哈密顿约束的其他替代量子化中被观测到 [212,213]，在空间平坦的 FLRW 宇宙中，大爆炸奇点再次得到解决。因此，反弹另一侧能否恢复经典宇宙现在存疑。尽管可以证明特定圈量子引力模型可以得到反弹演化 [214]，上述结果仍表明标准 LQC 可能无法涵盖 LQG 量子宇宙学部分的全部特性。除了这些修正 LQC 模型，研究 LQC 与 LQG 之间联系的其他策略还包括约化量子圈量子引力 [70]、群理论宇宙学 [71,72] 以及路径积分方法 [73]。值得注意的是，尽管带有 $\bar{\mu}$ 方案的改进动力学最初是在均匀各向同性 FLRW LQC 宇宙中发现的，但其有效性已经推广到其他时空和其他量子化方案。此外，$\bar{\mu}$ 方案也出现在约化量子圈量子引力和路径积分方法这类其他方法中。因此，作为目前已知均匀各向同性 LQC 宇宙学模型中唯一可行的方案，$\bar{\mu}$ 方案即使在更一般的设定下似乎仍然自洽。

Finally, although in this chapter we have mainly focused on the cosmological spacetimes, it is important to note that loop quantization is not only restricted to the FLRW universe. Numerous works have been done to apply loop quantization to the spherically symmetric spacetimes for the purpose of exploring the role of the non-perturbative quantum geometry effects in resolving the strong singularity encountered in the classical black hole spacetimes. Depending on the degrees of freedom that are to be quantized, the models of LQG black holes can be classified into two main categories. The models in the first category are quantized in the mini-superspace where symmetry reduction and homogeneity are assumed first at the classical level. In this case, the isometry between the Schwarzschild interior and the Kantowski-Sachs spacetime in cosmology is usually used, and one only needs to deal with finite degrees of freedom [215-220]. Here quantum difference equation has much richer structure than the isotropic LQC whose simulations also indicate singularity resolution [221]. On the other hand, the models in the second category only appeal to the spherical symmetry, and one needs to deal with the infinite degrees of freedom [222-227] which make them similar to the Gowdy models in cosmology. In addition to the vacuum spacetimes, investigations have also been extended to the cases which include some matter fields, such as the scalar fields and the dust fields [227-232]. In all these cases, similar to the big-bang singularity in cosmology, the curvature singularity at the center of black hole is resolved and replaced by a bounce. However, the physics beyond the bounce is model-dependent. The black hole can be glued to a white hole with the same/different mass or Nariai universe resulting from an emergent cosmological constant. Research in the LQG black holes is still full of actively ongoing subjects, and more astounding discoveries are expected in the near future.

最后，尽管本章我们主要聚焦于宇宙学时空，但需要注意的是，环量子化并不仅限于 FLRW 宇宙。已有大量研究将环量子化应用到球对称时空，以此探究非微扰量子几何效应在解决经典黑洞时空的强奇点问题中发挥的作用。根据待量子化的自由度，LQG 黑洞模型可主要分为两类。第一类模型是在迷你超空间中量子化，即先在经典层面假设对称性约化与均匀性。这类模型通常利用史瓦西内区与宇宙学中的坎托夫斯基-萨克斯时空之间的等距性，仅需要处理有限自由度 [215-220]。此处的量子差分方程比各向同性 LQC 的结构丰富得多，其数值模拟也证实奇点得到了解决 [221]。另一方面，第二类模型仅要求球对称，需要处理无穷多自由度 [222-227]，因此这类模型和宇宙学中的高迪模型性质相似。除真空时空外，相关研究也已拓展到包含标量场、尘埃场等物质场的情况 [227-232]。在所有这些情况中，和宇宙学的大爆炸奇点类似，黑洞中心的曲率奇点得到了解决，被一个反弹过程替代。不过，反弹之后的物理性质依赖于具体模型: 黑洞可以和质量相同/不同的白洞相连，也可以连接由涌现宇宙学常数得到的纳雷宇宙。LQG 黑洞的研究仍有大量活跃的方向，有望在不久的将来迎来更多令人惊喜的发现。

Cross-References

交叉引用

Hamiltonian Theory: Dynamics

哈密顿理论: 动力学

- Loop Quantum Cosmology: Relation Between Theory and Observations

  - 圈量子宇宙学: 理论与观测的关联

- Quantum Geometry and Black Holes

  - 量子几何与黑洞

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